

LARGE DEVIATION RATE FUNCTIONS FOR THE PARTITION FUNCTION IN A LOG-GAMMA DISTRIBUTED RANDOM POTENTIAL

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ABSTRACT. We study right tail large deviations of the logarithm of the partition function for directed lattice paths in i.i.d. random potentials. The main purpose is the derivation of explicit formulas for the 1+1-dimensional exactly solvable case with log-gamma distributed random weights. Along the way we establish some regularity results for this rate function for general distributions in arbitrary dimensions.

1. INTRODUCTION

We study a version of the model called *directed polymer in a random environment* where a fluctuating path is coupled with a random environment. This model was introduced in the statistical physics literature in [16] and early mathematically rigorous work followed in [3, 17]. We consider directed paths in the nonnegative orthant \mathbb{Z}_+^d of the d -dimensional integer lattice. The paths are allowed nearest-neighbor steps oriented along the coordinate axes. A random weight $\omega(\mathbf{u})$ is attached to each lattice point $\mathbf{u} \in \mathbb{Z}_+^d$. Together the weights form the *environment* $\omega = \{\omega(\mathbf{u}) : \mathbf{u} \in \mathbb{Z}_+^d\}$. The space of environments is denoted by Ω . \mathbb{P} is a probability measure on Ω under which the weights $\{\omega(\mathbf{u})\}$ are i.i.d. random variables.

For $\mathbf{v}, \mathbf{u} \in \mathbb{Z}_+^d$ such that $\mathbf{v} \leq \mathbf{u}$ (coordinatewise ordering) the set of admissible paths from \mathbf{v} to \mathbf{u} with $|\mathbf{u} - \mathbf{v}|_1 = m$ is

$$(1.1) \quad \Pi_{\mathbf{v}, \mathbf{u}} = \{x. = \{\mathbf{v} = x_0, x_1, \dots, x_m = \mathbf{u}\} : \forall k, x_k \in \mathbb{Z}_+^d \text{ and } x_{k+1} - x_k \in \{\mathbf{e}_i : 1 \leq i \leq d\}\}$$

where \mathbf{e}_i is the i -th standard basis vector of \mathbb{R}^d . The *point-to-point partition function* is

$$(1.2) \quad Z_{\mathbf{v}, \mathbf{u}} = \sum_{x. \in \Pi_{\mathbf{v}, \mathbf{u}}} e^{\sum_{j=1}^m \omega(x_j)}.$$

This is the normalization factor in the *quenched polymer distribution*

$$(1.3) \quad Q_{\mathbf{v}, \mathbf{u}}(x.) = Z_{\mathbf{v}, \mathbf{u}}^{-1} \prod_{j=1}^m e^{\omega(x_j)}$$

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which is a probability distribution on the paths in the set $\Pi_{\mathbf{v}, \mathbf{u}}$. When paths start at the origin ($\mathbf{v} = \mathbf{0}$) we drop \mathbf{v} from the notation; $Z_{\mathbf{u}} = Z_{\mathbf{0}, \mathbf{u}}$ and $\Pi_{\mathbf{u}} = \Pi_{\mathbf{0}, \mathbf{u}}$. Note that the weight at the starting point x_0 was not included in the sum in the exponent in (1.2). This makes no difference for the results. Sometimes it is convenient to include this weight and then we write $Z_{\mathbf{v}, \mathbf{u}}^{\square} = e^{\omega(\mathbf{v})} Z_{\mathbf{v}, \mathbf{u}}$ where the superscript \square reminds us that all weights in the rectangle are included.

In the polymer model one typically studies fluctuations of the path and fluctuations of $\log Z_{\mathbf{u}}$. This paper considers only $\log Z_{\mathbf{u}}$. Specifically our main object of interest is the right tail large deviation rate function

$$(1.4) \quad J_{\mathbf{u}}(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\}$$

where $\mathbf{u} \in \mathbb{R}_+^d$, $r \in \mathbb{R}$ and the floor of a vector is $\lfloor n\mathbf{y} \rfloor = (\lfloor ny_1 \rfloor, \lfloor ny_2 \rfloor, \dots, \lfloor ny_d \rfloor)$. This function J exists very generally for superadditivity reasons, and in Section 3 we establish some of its regularity properties.

The focus of the paper is an exactly solvable case where $d = 2$ and $-\omega(\mathbf{u})$ is log-gamma distributed. By “exactly solvable” we mean that special properties of the log-gamma case permit explicit computations, such as a formula for the limiting point-to-point free energy

$$(1.5) \quad p(\mathbf{y}) = \lim_{n \rightarrow \infty} n^{-1} \log Z_{\lfloor n\mathbf{y} \rfloor} \quad \mathbb{P} - a.s.$$

and fluctuation exponents [31]. In the same spirit, in this paper we compute explicit formulas for the rate function J and other related quantities in the context of the 1+1-dimensional log-gamma polymer.

One can also consider point-to-line partition functions over all directed paths of a fixed length. For $m \in \mathbb{N}$ the partition function is defined by

$$(1.6) \quad Z_m^{\text{line}} = \sum_{\mathbf{u} \in \mathbb{Z}_+^d : |\mathbf{u}|_1 = m} Z_{\mathbf{u}}.$$

Due to the $n^{-1} \log$ in front, in the results we look at Z_m^{line} behaves like the maximal $Z_{\mathbf{u}}$ over $|\mathbf{u}|_1 = m$.

Some comments are in order.

There are currently three known exactly solvable directed polymer models, all in 1+1 dimensions: The two with a discrete aspect are (i) the log-gamma model introduced in [31], (ii) a model introduced in [27] where the random environment is a collection of Brownian motions. Some fluctuation exponents were derived for the second model in [32], and it has been further studied in [26] via a connection with the quantum Toda lattice. This Brownian model possesses structures similar to those in the log-gamma model, so we expect that the results of the present paper could be reproduced for the Brownian model.

The third exactly solvable model is the continuum directed random polymer [1] that is expected to be a universal scaling limit for a large class of polymer models (see [10] for a recent review).

Usually the directed lattice polymer model is placed in a space-time picture where the paths are oriented in the time direction. (See articles and lectures [5, 6, 8, 13] for recent results and reviews of the general case.) In two dimensions (1 time + 1 space dimension) the space-time picture is the same as our purely spatial picture, up to a 45° rotation of

the lattice and a change of lattice indices. The temporal aspect is not really present in our work. So we have not separated a time dimension but simply regard the paths as directed lattice paths.

Another standard feature of directed polymers that we have omitted is the inverse temperature parameter $\beta \in (0, \infty)$ that usually appears as a multiplicative constant in front of the weights: $Z_{\mathbf{v}, \mathbf{u}}^\beta = \sum_{x \in \Pi_{\mathbf{v}, \mathbf{u}}} \exp\{\beta \sum_{j=1}^m \omega(x_j)\}$. For a fixed weight distribution, β modulates the strength of the coupling between the walk and the environment. It is known that in dimension 1+3 and higher there can be a phase transition. By contrast, in low dimensions (1+1 and 1+2) the model is in the so-called strong coupling regime for all $0 < \beta < \infty$ [7, 21]. The β parameter plays no role in the present work and has a fixed value $\beta = 1$. This is the unique β value that turns the log-gamma model into an exactly solvable model.

The techniques of the current paper are entirely probabilistic and rely on the stationary version of the log-gamma model. It can be expected that as a combinatorial approach to this model is fully developed [11], more complete results and alternative proofs for the present results can be found.

Earlier literature. Precise large deviation rate functions for $\log Z$ in the case of directed polymers have not been derived in the past. The strongest concentration inequalities can be found in recent references [9, 22, 33]. The normalization of the left tail varies with the distribution of the weights as demonstrated by [2] but the right tails have the same normalization n . [4] has some bounds on the left tail of $\log Z$ in Gaussian environments in dimensions 1 + 3 and higher and for small enough β . Similar bounds were proved later in [24] for bounded environments using concentration inequalities for product measures.

For the exactly solvable zero-temperature models (that is, last passage percolation models) large deviation principles have been proved. For the longest increasing path among planar Poisson points, an LDP for the length resulted from a combination of articles [14, 20, 23, 30]. These results came before the advent of determinantal techniques. For the corner growth model with geometric and exponential weights [18] derived an LDP in addition to the Tracy-Widom limit. An earlier right tail LDP appeared in [29].

Notation. We collect some notation and conventions here for easy reference. \mathbb{N} is for positive integers, \mathbb{Z}_+ for nonnegative integer, \mathbb{R}_+ for nonnegative real numbers and \mathbb{R}_+^d is the set of all vectors with nonnegative real coordinates. Vector notation: elements of \mathbb{R}^d and \mathbb{Z}^d are $\mathbf{v} = (v_1, v_2, \dots, v_d)$. Coordinatewise ordering $\mathbf{v} \leq \mathbf{u}$ means $v_1 \leq u_1, v_2 \leq u_2, \dots, v_d \leq u_d$. Particular vectors: $\mathbf{1} = (1, 1, \dots, 1)$ and $\mathbf{0} = (0, 0, \dots, 0)$. $\lfloor \mathbf{y} \rfloor = (\lfloor y_1 \rfloor, \lfloor y_2 \rfloor, \dots, \lfloor y_d \rfloor)$ where $\lfloor y \rfloor = \max\{n \in \mathbb{Z} : n \leq y\}$ is the integer part of $y \in \mathbb{R}$. The ℓ^1 norm on \mathbb{R}^d is $|\mathbf{v}|_1 = |v_1| + \dots + |v_d|$.

The convex dual of a function $f : \mathbb{R} \rightarrow (-\infty, \infty]$ is $f^*(y) = \sup_{x \in \mathbb{R}} \{xy - f(x)\}$, and $f = f^{**}$ iff f is convex and lower semicontinuous. We refer to [28] for basic convex analysis.

The partition function Z does not include the weight of the initial point of the paths, while Z^\square does. In 2 dimensions we write $Z_{m,n} = Z_{(m,n)}$.

The usual gamma function is $\Gamma(\mu) = \int_0^\infty x^{\mu-1} e^{-x} dx$ for $\mu > 0$. The digamma and trigamma functions are $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_1 = \Psi_0'$. On $(0, \infty)$ Ψ_0 is increasing and concave and Ψ_1 decreasing, positive and convex, with $-\Psi_0(0+) = \Psi_1(0+) = \infty$.

2. LARGE DEVIATIONS FOR THE LOG-GAMMA MODEL

2.1. The log-gamma model with i.i.d. weights. In this section we specialize to $d = 2$ dimensions and the log-gamma distributed weights. Fix a positive real parameter μ . This parameter remains fixed through this entire section, and hence is omitted from most notation. In the log-gamma case we prefer to switch to multiplicative variables. So the weight at point $(i, j) \in \mathbb{Z}_+^2$ is $Y_{i,j} = e^{\omega(i,j)}$ where the reciprocal Y^{-1} has $\text{Gamma}(\mu)$ distribution. Explicitly,

$$(2.1) \quad \mathbb{P}\{Y^{-1} \geq s\} = \Gamma(\mu)^{-1} \int_s^\infty x^{\mu-1} e^{-x} dx \quad \text{for } s \in \mathbb{R}_+.$$

As above, we write Y for a generic random variable distributed as $Y_{i,j}$. The digamma and trigamma functions give the mean and variance: $\mathbb{E}(\log Y) = -\Psi_0(\mu)$ and $\text{Var}(\log Y) = \Psi_1(\mu)$.

The logarithmic moment generating function (l.m.g.f.) of $\omega = \log Y$ is

$$(2.2) \quad M_\mu(\xi) = \log \mathbb{E}(e^{\xi \log Y}) = \begin{cases} \log \Gamma(\mu - \xi) - \log \Gamma(\mu), & \xi \in (-\infty, \mu) \\ \infty, & \xi \in [\mu, \infty). \end{cases}$$

The point-to-point partition function for directed paths from $(0, 0)$ to (m, n) is

$$(2.3) \quad Z_{m,n} = \sum_{x \in \Pi_{(m,n)}} \prod_{j=1}^{m+n} Y_{x_j}.$$

Note that we simplified notation by dropping the parentheses: $Z_{m,n} = Z_{(m,n)}$. For $(s, t) \in \mathbb{R}_+^2$ the limiting free energy density exists by superadditivity:

$$(2.4) \quad p(s, t) = \lim_{n \rightarrow \infty} n^{-1} \log Z_{[ns], [nt]} \quad \mathbb{P}\text{-a.s.}$$

The limit is a finite constant. We begin by giving its exact value.

THEOREM 2.1. *For $(s, t) \in \mathbb{R}_+^2$ and $\mu \in (0, \infty)$, the limiting free energy density (2.4) is given by*

$$(2.5) \quad p(s, t) = \inf_{0 < \rho < \mu} \{-s\Psi_0(\rho) - t\Psi_0(\mu - \rho)\}.$$

The value $p(s, t)$ was already derived in [31] but the proof was buried among estimates for fluctuation exponents. In Section 4 we sketch an elementary approach that utilizes special features of the log-gamma model. For the other explicitly solvable 1+1 dimensional polymer with Brownian environment, [25] computed the limiting free energy with a very different large deviation approach.

The next result is a large deviation principle (LDP) for $\log Z_{[ns], [nt]}$ under normalization n . The rate function is

$$(2.6) \quad I_{s,t}(r) = \begin{cases} \sup_{\xi \in [0, \mu]} \{r\xi - \inf_{\theta \in (\xi, \mu)} (tM_\theta(\xi) - sM_{\mu-\theta}(-\xi))\}, & r \geq p(s, t) \\ \infty, & r < p(s, t). \end{cases}$$

On the boundary ($s = 0$ or $t = 0$) the result reduces to i.i.d. large deviations so we only consider (s, t) in the interior of the quadrant.

THEOREM 2.2. *Let $Y^{-1} \sim \text{Gamma}(\mu)$ as in (2.1) and $(s, t) \in (0, \infty)^2$. Then the distributions of $n^{-1} \log Z_{[ns], [nt]}$ satisfy a LDP with normalization n and rate function $I_{s,t}$. Explicitly, these bounds hold for any open set G and any closed set F in \mathbb{R} :*

$$(2.7) \quad \overline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{n^{-1} \log Z_{[ns], [nt]} \in F\} \leq - \inf_{r \in F} I_{s,t}(r),$$

and

$$(2.8) \quad \underline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{n^{-1} \log Z_{[ns], [nt]} \in G\} \geq - \inf_{r \in G} I_{s,t}(r).$$

On $[p(s, t), \infty)$ the rate function $I_{s,t}$ is finite, strictly increasing, continuous and convex. In particular, the unique zero of $I_{s,t}(r)$ is at $r = p(s, t)$. The right tail rate defined in (1.4) is given by

$$(2.9) \quad J_{s,t}(r) = \begin{cases} 0, & r \in (-\infty, p(s, t)] \\ I_{s,t}(r), & r \in [p(s, t), \infty). \end{cases}$$

Remark 2.3. From a computational point of view, the solution to the variational problem in (2.6) can be computed by

$$I_{s,t}(r) = \sup_{0 < \theta < \mu} \{f_r(\theta) - \inf_{0 < z \leq \theta} f_r(z)\} = f_r(\theta_2) - f_r(\theta_1),$$

where

$$f_r(\theta) = r\theta + t \log \Gamma(\theta) - s \log \Gamma(\mu - \theta),$$

and for any $r > p(s, t)$, $0 < \theta_1 < \theta_2 < \mu$ are the solutions to the equation $\frac{d}{d\theta} f_r(\theta) = 0$. (See Fig.1.) This again implies that the rate function is strictly positive as long as $r > p(s, t)$.

Remark 2.4. We do not address the precise large deviations in the left tail, that is, in the range $r < p(s, t)$. We expect the correct normalization to be n^2 . (Personal communication from I. Ben-Ari.) Presently we do not have a technique for computing the rate function in that regime. We include the trivial part $I_{s,t}(r) = \infty$ for $r < p(s, t)$ in the theorem so that we can compute the limiting l.m.g.f. by a straightforward application of Varadhan's theorem.

Define for $\xi \in \mathbb{R}$

$$(2.10) \quad \Lambda_{s,t}(\xi) = \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns], [nt]}}.$$

COROLLARY 2.5. *Let $\xi \in \mathbb{R}$. Then the limit in (2.10) exists and is given by*

$$(2.11) \quad \Lambda_{s,t}(\xi) = I_{s,t}^*(\xi) = \begin{cases} p(s, t)\xi, & \xi < 0 \\ \inf_{\theta \in (\xi, \mu)} \{tM_\theta(\xi) - sM_{\mu-\theta}(-\xi)\}, & 0 \leq \xi < \mu \\ \infty, & \xi \geq \mu. \end{cases}$$

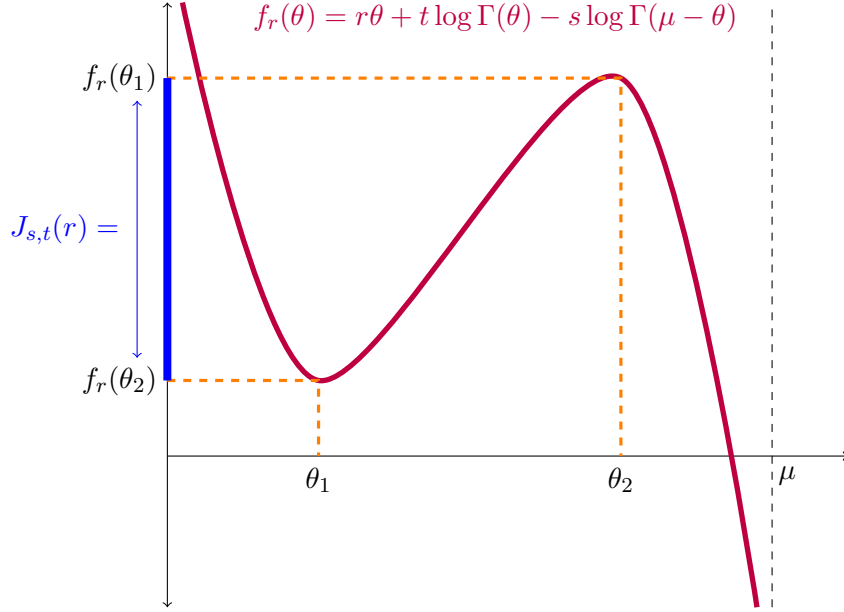


FIGURE 1. Graphical representation of the solution to the variational problem (2.6) that gives the rate function $J_{s,t}(r) = f_r(\theta_2) - f_r(\theta_1)$. The curve $f_r(\theta)$ has the same general shape as long as $r > p(s, t)$.

Remark 2.6. Symmetry of $\Lambda_{s,t}$ in (s, t) is clear from (2.10) but not immediately obvious in the $0 \leq \xi < \mu$ case of (2.11). It turns out that if $s \leq t$ the infimum is achieved at a unique $\theta_0 \in [(\mu + \xi)/2, \mu)$, and then for $\Lambda_{t,s}(\xi)$ the same infimum is uniquely achieved at $\theta_1 = \mu + \xi - \theta_0 \in (\xi, (\mu + \xi)/2]$. In the case $s = t$ a simple formula arises: $\Lambda_{t,t}(\xi) = 2t(\log \Gamma(\frac{\mu - \xi}{2}) - \log \Gamma(\frac{\mu + \xi}{2}))$.

Remark 2.7. The first case of (2.6) gives $I_{s,t}$ as the dual $\Lambda_{s,t}^*$, and the reader may wonder whether this is the logic of the proof of the LDP. It is not, for we have no direct way to compute $\Lambda_{s,t}$. Instead, Theorem 2.2 is first proved in an indirect manner via the stationary model described in the next subsection, and then $\Lambda_{s,t}$ is derived by Varadhan's theorem.

Let us also record the result for the point-to-line case. It behaves like the point-to-point case along the diagonal.

COROLLARY 2.8. *Let $Y^{-1} \sim \text{Gamma}(\mu)$ as in (2.1) and $s > 0$. Then the distributions of $\log Z_{[ns]}^{\text{line}}$ satisfy a LDP with normalization n and rate function $I_{s/2, s/2}$.*

Remark 2.9. For $\varepsilon > 0$ and $r = p(s, t) + \varepsilon$, one can show after some calculus that there exists a non zero constant $C = C_{s,t}(\mu)$ so that

$$I_{s,t}(r) = C\varepsilon^{3/2} + o(\varepsilon^{3/2}).$$

This suggests that $\text{Var}(\log Z_{[ns], [nt]})$ is of order $n^{2/3}$. Rigorous upper bounds on the moments $\mathbb{E}|\log Z_{[ns], [nt]} - np(s, t)|^p$ for $1 \leq p < 3/2$ can be found in [31], Theorem 2.4.

We computed the precise value of the constant C for the point-to-line rate function,

$$(2.12) \quad I_{1,1}(r) = \frac{4}{3} \frac{1}{\sqrt{|\Psi_2(\mu/2)|}} \varepsilon^{3/2} + o(\varepsilon^{3/2}),$$

where $\Psi_2 = \Psi_0''$.

2.2. The stationary log-gamma model. Next we consider the log-gamma model in a stationary situation that is special to this choice of distribution. Working with the stationary case is the key to explicit computations, including all the previous results, and provides some explanation for the formulas that arose for $I_{s,t}$ and $\Lambda_{s,t}$ in (2.6) and (2.11).

The stationary model is created by appropriately altering the distributions of the weights on the boundaries of the quadrant \mathbb{Z}_+^2 . We continue to use the parameter $\mu \in (0, \infty)$ fixed at the beginning of this section, and we introduce a second parameter $\theta \in (0, \mu)$. Let the collection of independent weights $\{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\}$ have these marginal distributions:

$$(2.13) \quad U_{i,0}^{-1} \sim \text{Gamma}(\theta), \quad V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta), \quad \text{and} \quad Y_{i,j}^{-1} \sim \text{Gamma}(\mu).$$

Define the partition function $Z_{m,n}^{(\theta)}$ by (2.3) with the following weights: at the origin $Y_{0,0} = 1$, on the x -axis $Y_{i,0} = U_{i,0}$, on the y -axis $Y_{0,j} = V_{0,j}$, and in the bulk the weights $\{Y_{i,j} : i, j \in \mathbb{N}\}$ are i.i.d. $\text{Gamma}(\mu)^{-1}$ as before. Equivalently, we can decompose the stationary partition function $Z_{m,n}^{(\theta)}$ according to the exit point of the path from the boundary:

$$(2.14) \quad Z_{m,n}^{(\theta)} = \sum_{k=1}^m \left(\prod_{i=1}^k U_{i,0} \right) Z_{(k,1),(m,n)}^{\square} + \sum_{\ell=1}^n \left(\prod_{j=1}^{\ell} V_{0,j} \right) Z_{(1,\ell),(m,n)}^{\square}.$$

The symbols $U_{i,0}$ and $V_{0,j}$ were at first introduced for the boundary weights to highlight the change of distribution. Next let us define for all $(i, j) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$

$$(2.15) \quad U_{i,j} = \frac{Z_{i,j}^{(\theta)}}{Z_{i-1,j}^{(\theta)}} \quad \text{and} \quad V_{i,j} = \frac{Z_{i,j}^{(\theta)}}{Z_{i,j-1}^{(\theta)}}.$$

Note that this property was already built into the boundaries because for example $Z_{i,0}^{(\theta)} = U_{1,0} \cdots U_{i,0}$. The key result that allows explicit calculations for this model is the following.

PROPOSITION 2.10. *For each $(i, j) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$ we have the following marginal distributions: $U_{i,j}^{-1} \sim \text{Gamma}(\theta)$ and $V_{i,j}^{-1} \sim \text{Gamma}(\mu - \theta)$. For any fixed $n \in \mathbb{Z}_+$, the variables $\{U_{i,n} : i \in \mathbb{N}\}$ are i.i.d., and for any fixed $m \in \mathbb{Z}_+$, the variables $\{V_{m,j} : j \in \mathbb{N}\}$ are i.i.d.*

This is a special case of Theorem 3.3 in [31], where the independence of these weights along more general down-right lattice paths is established. Proposition 2.10 is the only result from [31] that we use. It follows in an elementary fashion from the properties of the gamma distribution.

As an immediate application we can write

$$(2.16) \quad n^{-1} \log Z_{[ns],[nt]}^{(\theta)} = n^{-1} \sum_{j=1}^{[nt]} \log V_{0,j} + n^{-1} \sum_{i=1}^{[ns]} \log U_{i,[nt]}$$

as a sum of two sums of i.i.d. variables, and from this compute

$$(2.17) \quad \mathbb{E}(\log Z_{m,n}^{(\theta)}) = m\mathbb{E}(\log U) + n\mathbb{E}(\log V) = -m\Psi_0(\theta) - n\Psi_0(\mu - \theta)$$

and obtain the law of large numbers:

$$(2.18) \quad n^{-1} \log Z_{[ns],[nt]}^{(\theta)} \rightarrow p^{(\theta)}(s, t) = -s\Psi_0(\theta) - t\Psi_0(\mu - \theta) \quad \mathbb{P}\text{-a.s.}$$

Note that the two sums on the right of (2.16) are not independent of each other. In fact, they are so strongly negatively correlated that the variance of their sum is of order $n^{2/3}$ [31]. Comparison of (2.5) and (2.18) reveals a variational principle at work: $p(s, t)$ is the minimal free energy of a stationary system with bulk parameter μ .

Instead of the right tail large deviation rate function we give the asymptotic l.m.g.f. in the next result. Define

$$(2.19) \quad \Lambda_{\theta,(s,t)}(\xi) = \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta)}}.$$

THEOREM 2.11. *Let $s, t \geq 0$ and $0 < \theta < \mu$. Then the limit in (2.19) exists for $\xi \geq 0$ and is given by*

$$(2.20) \quad \Lambda_{\theta,(s,t)}(\xi) = \begin{cases} \max \{ sM_\theta(\xi) - tM_{\mu-\theta}(-\xi), tM_{\mu-\theta}(\xi) - sM_\theta(-\xi) \}, & 0 \leq \xi < \theta \wedge (\mu - \theta) \\ \infty, & \xi \geq \theta \wedge (\mu - \theta). \end{cases}$$

Remark 2.12. Let the parameters $0 < \theta < \mu$ be given. The *characteristic direction* is the choice

$$(2.21) \quad (s, t) = c(\Psi_1(\mu - \theta), \Psi_1(\theta)) \quad \text{for a constant } c > 0.$$

With this choice the variance of $\log Z_{[ns],[nt]}^{(\theta)}$ is of order $n^{2/3}$, while in other directions the fluctuations of $\log Z_{[ns],[nt]}^{(\theta)}$ have order of magnitude $n^{1/2}$ and they are asymptotically Gaussian [31]. By this token, we would expect the large deviations in the characteristic situation to be unusual, while in the off-characteristic directions we would expect the more typical large deviations of order e^{-n} in both tails. In Lemma 4.2(b) we give a bound on the left tail that indicates superexponential decay under (2.21). This also implies that if (2.21) holds, then formula (2.20) can be complemented with the case $\Lambda_{\theta,(s,t)}(\xi) = p^{(\theta)}(s, t)\xi$ for $\xi \leq 0$. Presently we do not have further information about these large deviations.

Remark 2.13. If the two sums in (2.16) were independent we would have $\Lambda_{\theta,(s,t)}(\xi) = sM_\theta(\xi) + tM_{\mu-\theta}(\xi)$. Obviously (2.20) reflects the strong negative correlation of these sums, but currently we do not have a good explanation (besides the proof!) for the formula that arises.

The maximum in (2.20) comes from the choice of the first step of the path: either horizontal or vertical. Corresponding to this choice, define partition functions

$$(2.22) \quad Z_{[ns],[nt]}^{(\theta),\text{hor}} = \sum_{k=1}^{[ns]} \left(\prod_{i=1}^k U_{i,0} \right) Z_{(k,1),([ns],[nt])}^\square$$

and

$$(2.23) \quad Z_{[ns],[nt]}^{(\theta),\text{ver}} = \sum_{\ell=1}^{\lfloor nt \rfloor} \left(\prod_{j=1}^{\ell} V_{0,j} \right) Z_{(1,\ell),([ns],[nt])}^{\square},$$

together with l.m.g.f.'s

$$(2.24) \quad \Lambda_{\theta,(s,t)}^{\text{hor}}(\xi) = \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{hor}}}$$

and

$$(2.25) \quad \Lambda_{\theta,(s,t)}^{\text{ver}}(\xi) = \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{ver}}}.$$

Then $Z_{[ns],[nt]}^{(\theta)} = Z_{[ns],[nt]}^{(\theta),\text{hor}} + Z_{[ns],[nt]}^{(\theta),\text{ver}}$ leads to

$$(2.26) \quad \Lambda_{\theta,(s,t)}(\xi) = \Lambda_{\theta,(s,t)}^{\text{hor}}(\xi) \vee \Lambda_{\theta,(s,t)}^{\text{ver}}(\xi)$$

which is the starting point for the proof of (2.20).

The horizontal and vertical partition functions are in some sense between the stationary one and the one from (2.3) with i.i.d. weights. It turns out that these intermediate partition functions behave either like the stationary one or like the i.i.d. one, with a sharp transition in between, and this holds both at the level of the limiting free energy density and the l.m.g.f. Let us focus on the horizontal case, the vertical case being the same after the swap $s \leftrightarrow t$ and $\theta \leftrightarrow \mu - \theta$.

Qualitatively, with t fixed, when s is large $Z_{[ns],[nt]}^{(\theta),\text{hor}}$ behaves like $Z_{[ns],[nt]}^{(\theta)}$, and when s is small $Z_{[ns],[nt]}^{(\theta),\text{hor}}$ behaves like $Z_{[ns],[nt]}$ from (2.3). Here are the conditions for the transitions:

$$(2.27) \quad s\Psi_1(\theta) \geq t\Psi_1(\mu - \theta)$$

and

$$(2.28) \quad s(\Psi_0(\theta) - \Psi_0(\theta - \xi)) \geq t(\Psi_0(\mu - \theta + \xi) - \Psi_0(\mu - \theta)).$$

By the concavity of Ψ_0 and the fact that $\Psi_1 = \Psi'_0$, (2.27) implies (2.28) for all $\xi \geq 0$. Assuming the limit exists for the moment, define

$$(2.29) \quad p^{(\theta),\text{hor}}(s,t) = \lim_{n \rightarrow \infty} n^{-1} \log Z_{[ns],[nt]}^{(\theta),\text{hor}}.$$

In this next theorem the functions $p(s,t)$ and $\Lambda_{s,t}(\xi)$ are the ones defined by (2.5) and (2.11).

THEOREM 2.14. *Let $s, t \geq 0$, $0 < \theta < \mu$ and $0 \leq \xi < \theta$.*

(a) *The limit in (2.29) exists and is given by*

$$(2.30) \quad p^{(\theta),\text{hor}}(s,t) = \begin{cases} p^{(\theta)}(s,t), & \text{if (2.27) holds} \\ p(s,t), & \text{if (2.27) fails.} \end{cases}$$

(b) *The limit in (2.24) exists and is given by*

$$(2.31) \quad \Lambda_{\theta,(s,t)}^{\text{hor}}(\xi) = \begin{cases} sM_{\theta}(\xi) - tM_{\mu-\theta}(-\xi), & \text{if (2.28) holds} \\ \Lambda_{s,t}(\xi), & \text{if (2.28) fails.} \end{cases}$$

Remark 2.15. We saw in (2.5) that the limiting free energy $p(s, t)$ of the i.i.d. model is the minimal free energy of the stationary models with the same bulk parameter μ . This link does *not* extend to the l.m.g.f.'s: for $0 < \xi < \mu$, $\Lambda_{s,t}(\xi) < \Lambda_{\theta,(s,t)}(\xi)$ for all $\theta \in (0, \mu)$. We observe this at the end of the proof of Theorem 2.11 in Section 5.

3. THE RIGHT TAIL RATE FUNCTION IN THE GENERAL CASE

The proofs of the results for the log-gamma model utilize regularity properties of the rate function J of (1.4). These properties can be proved in some degree of generality, and we do so in this section. So now we consider

$$(3.1) \quad Z_{\mathbf{u}} = \sum_{x \in \Pi_{\mathbf{u}}} e^{\sum_{j=1}^{|\mathbf{u}|_1} \omega(x_j)}$$

as defined in the Introduction, with $\mathbf{u} \in \mathbb{Z}_+^d$, general $d \geq 2$, and general i.i.d. weights $\{\omega(\mathbf{u})\}$.

We assume

$$(3.2) \quad \exists \xi > 0 \text{ such that } \mathbb{E}(e^{\xi|\omega(\mathbf{u})|}) < \infty.$$

This guarantees the existence of a Cramér large deviation rate function defined by

$$(3.3) \quad I(r) = -\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \left\{ n^{-1} \sum_{i=1}^n \omega(\mathbf{u}_i) \in (r - \varepsilon, r + \varepsilon) \right\}.$$

(Above $\{\mathbf{u}_j\}$ are any distinct lattice points.) We state first the existence theorem for the limiting point-to-point free energy density. We omit the proof because similar superadditive and approximation arguments appear elsewhere in our paper, and refer to [15]. Let us also point out that assumption (3.2) is unnecessarily strong for this existence result but our objective here is not to optimize on this point.

THEOREM 3.1. *Assume (3.2). There exists an event $\Omega_0 \subseteq \Omega$ of full \mathbb{P} -probability on which the convergence*

$$(3.4) \quad p(\mathbf{y}) = \lim_{n \rightarrow \infty} n^{-1} \log Z_{\lfloor n\mathbf{y} \rfloor}$$

happens simultaneously for all $\mathbf{y} \in \mathbb{R}_+^d$. Limit (3.4) holds also in $L^1(\mathbb{P})$. As a function of \mathbf{y} , p is concave and continuous on \mathbb{R}_+^d .

Next the right-tail LDP. To avoid issues of vanishing probabilities and infinite values of the rate, we make this further assumption:

$$(3.5) \quad \forall r < \infty : \mathbb{P}\{\omega(\mathbf{0}) > r\} > 0.$$

THEOREM 3.2. *Assume (3.2) and (3.5). Then for $\mathbf{u} \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$ and $r \in \mathbb{R}$ this \mathbb{R}_+ -valued limit exists:*

$$(3.6) \quad J_{\mathbf{u}}(r) = -\lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\}.$$

As a function of (\mathbf{u}, r) , J is convex and continuous on $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$. $J_{\mathbf{u}}(r) = 0$ iff $r \leq p(\mathbf{u})$.

Let us also remark that the weight $\omega(\mathbf{0})$ at the origin is immaterial: the limit is the same for Z^\square , so for $\mathbf{u} \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}$ and $r \in \mathbb{R}$,

$$(3.7) \quad J_{\mathbf{u}}(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor}^\square \geq nr\}.$$

We observe this at the end of the proof of Theorem 3.2.

With a further assumption on the Cramér rate function of the weight distribution defined in (3.3) we can extend the continuity of $J_{\mathbf{u}}$ to $\mathbf{u} = \mathbf{0}$:

$$(3.8) \quad \alpha_\infty = \lim_{x \nearrow \infty} x^{-1} I(x) < \infty.$$

(3.5) is equivalent to requiring that $I(x) < \infty$ for all large enough x , so of course (3.8) requires (3.5). The constant α_∞ is the limiting slope of I at ∞ which exists by convexity. When assumption (3.8) is in force we define

$$(3.9) \quad J_{\mathbf{0}}(r) = \begin{cases} 0, & r \leq 0, \\ \alpha_\infty r, & r \geq 0. \end{cases}$$

THEOREM 3.3. *Under assumptions (3.2) and (3.8), and with $J_{\mathbf{0}}$ defined by (3.9), $J_{\mathbf{u}}(r)$ is finite and continuous on $\mathbb{R}_+^d \times \mathbb{R}$.*

Remark 3.4. Assumption (3.8) is in particular valid for the log-gamma model. For $Y^{-1} \sim \text{Gamma}(\mu)$ the Cramér rate function for $\omega = \log Y$ is

$$(3.10) \quad I_\mu(r) = -r\Psi_0^{-1}(-r) - \log \Gamma(\Psi_0^{-1}(-r)) + \mu r + \log \Gamma(\mu), \quad r \in \mathbb{R}.$$

The limiting slope on the right is $\alpha_\infty = \mu$, while the limiting slope on the left would be $\lim_{r \rightarrow -\infty} I'(r) = -\infty$. In this case $J_{\mathbf{0}}(r)$ is also the “rate function” for the single weight at the origin:

$$(3.11) \quad J_{\mathbf{0}}(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Y \geq nr\}.$$

The remainder of this section proves Theorems 3.2 and 3.3, and then we prove two further lemmas for later use.

Proof of Theorem 3.2. For $m, n \in \mathbb{R}_+$ let $\mathbf{x}_{m,n} \in \{0, 1\}^d$ so that $\lfloor (m+n)\mathbf{u} \rfloor = \lfloor m\mathbf{u} \rfloor + \lfloor n\mathbf{u} \rfloor + \mathbf{x}_{m,n}$. By superadditivity, independence and shift invariance

$$(3.12) \quad \begin{aligned} & \mathbb{P}\{\log Z_{\lfloor (m+n)\mathbf{u} \rfloor} \geq (m+n)r\} \\ & \geq \mathbb{P}\{\log Z_{\lfloor m\mathbf{u} \rfloor} \geq mr\} \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\} \mathbb{P}\{\log Z_{\mathbf{x}_{m,n}} \geq 0\}. \end{aligned}$$

By assumption (3.5) there is a uniform lower bound $\mathbb{P}\{\log Z_{\mathbf{x}_{m,n}} \geq 0\} \geq \rho > 0$. Thus $t(n) = \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\}$ is superadditive with a small uniformly bounded correction. Assumption (3.5) implies that $t(n) > -\infty$ for all $n \geq n_0$. Consequently by superadditivity the rate function

$$(3.13) \quad J_{\mathbf{u}}(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\}$$

exists for $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}_+^d$ and $r \in \mathbb{R}$. The limit in (3.13) holds also as $n \rightarrow \infty$ through real values, not just integers.

Similarly we get convexity of J in (\mathbf{u}, r) . Let $\lambda \in (0, 1)$ and assume $(\mathbf{u}, r) = \lambda(\mathbf{u}_1, r_1) + (1 - \lambda)(\mathbf{u}_2, r_2)$. Then

$$\begin{aligned} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\} &\geq \lambda(\lambda n)^{-1} \log \mathbb{P}\{\log Z_{\lfloor n\lambda\mathbf{u}_1 \rfloor} \geq n\lambda r_1\} \\ &\quad + (1 - \lambda)((1 - \lambda)n)^{-1} \log \mathbb{P}\{\log Z_{\lfloor n(1 - \lambda)\mathbf{u}_2 \rfloor} \geq n(1 - \lambda)r_2\} + o(1) \end{aligned}$$

and letting $n \rightarrow \infty$ gives

$$(3.14) \quad J_{\mathbf{u}}(r) \leq \lambda J_{\mathbf{u}_1}(r_1) + (1 - \lambda) J_{\mathbf{u}_2}(r_2).$$

Finiteness of J follows from (3.5), so now we know J to be a finite, convex function on $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$. This implies that J is continuous in the interior of $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$ and upper semicontinuous on the whole set $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$ [28, Thm. 10.1 and 10.2].

The law of large numbers for the free energy implies $J_{\mathbf{u}}(r) = 0$ for $r < p(\mathbf{u})$ and then by continuity for $r \leq p(\mathbf{u})$. With a minor adaptation of [9, Prop. 3.1(b)] we get a concentration inequality: given \mathbf{u} , for $\varepsilon > 0$ there exists a constant $c > 0$ such that

$$(3.15) \quad \mathbb{P}\{|\log Z_{\lfloor n\mathbf{u} \rfloor} - \mathbb{E} \log Z_{\lfloor n\mathbf{u} \rfloor}| \geq n\varepsilon\} \leq 2 \exp(-c\varepsilon^2 n) \quad \text{for all } n \in \mathbb{N}.$$

Since $n^{-1} \mathbb{E} \log Z_{\lfloor n\mathbf{u} \rfloor} \rightarrow p(\mathbf{u})$, this implies that $J_{\mathbf{u}}(r) > 0$ for $r > p(\mathbf{u})$.

We do a coupling proof for lower semicontinuity. Let $(\mathbf{u}, r) \rightarrow (\mathbf{v}, s)$ in $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$. If each coordinate $v_i > 0$ then we have continuity $J_{\mathbf{u}}(r) \rightarrow J_{\mathbf{v}}(s)$ because convexity already gives continuity in the interior. Thus we may assume that some coordinates of \mathbf{v} are zero. Since coordinates can be permuted without changing J , let us assume that $\mathbf{v} = (v_1, v_2, \dots, v_k, 0, \dots, 0)$ for a fixed $1 \leq k < d$ where $v_1, \dots, v_k > 0$. If eventually \mathbf{u} is also of the form $\mathbf{u} = (u_1, u_2, \dots, u_k, 0, \dots, 0)$ for the same k then we are done by convexity-implied continuity again, this time in the interior of $(\mathbb{R}_+^k \setminus \{\mathbf{0}\}) \times \mathbb{R}$.

The remaining case is the one where $u_1, \dots, u_k > 0$ and $(u_{k+1}, \dots, u_d) \rightarrow \mathbf{0}$. We develop a family of couplings that eliminates these $d - k$ last coordinates one by one, starting with u_d , and puts us back in the interior case with continuity. Denote a lower-dimensional projection by $\mathbf{u}_{1,k} = (u_1, u_2, \dots, u_k)$.

The set of paths $\Pi_{\lfloor n\mathbf{u} \rfloor}$ is decomposed according to the locations of the $\lfloor nu_d \rfloor$ unit jumps in the \mathbf{e}_d -direction. The projections of these locations form a vector π from the set

$$\Lambda_{\lfloor n\mathbf{u} \rfloor} = \left\{ \pi = \{\mathbf{x}^i\}_{i=0}^{\lfloor nu_d \rfloor + 1} \in (\mathbb{Z}_+^{d-1})^{\lfloor nu_d \rfloor + 2} : \mathbf{0} = \mathbf{x}^0 \leq \mathbf{x}^1 \leq \dots \leq \mathbf{x}^{\lfloor nu_d \rfloor + 1} = \lfloor n\mathbf{u}_{1,d-1} \rfloor \right\}.$$

The partition function then decomposes according to these jump locations:

$$(3.16) \quad Z_{\lfloor n\mathbf{u} \rfloor} = \sum_{\pi \in \Lambda_{\lfloor n\mathbf{u} \rfloor}} Z_{(\mathbf{0}, 0), (\mathbf{x}^1, 0)} \prod_{i=1}^{\lfloor nu_d \rfloor} Z_{(\mathbf{x}^i, i), (\mathbf{x}^{i+1}, i)}^{\square} \equiv \sum_{\pi \in \Lambda_{\lfloor n\mathbf{u} \rfloor}} Z_{\pi}$$

where the last equality defines the $d - 1$ -dimensional partition functions Z_{π} .

For a fixed π , define a new environment $\tilde{\omega}$ indexed by \mathbb{Z}_+^{d-1} with this recipe:

- (i) For $0 \leq i \leq \lfloor nu_d \rfloor$: for $\mathbf{y} \in \mathbb{Z}_+^{d-1}$ such that $\mathbf{x}^i \leq \mathbf{y} \leq \mathbf{x}^{i+1}$ but $\mathbf{y} \neq \mathbf{x}^i$, set $\tilde{\omega}(\mathbf{y}) = \omega(\mathbf{y}, i)$.
- (ii) $\tilde{\omega}(\mathbf{0}) = \omega(\mathbf{0}, 0)$ and for $1 \leq i \leq \lfloor nu_d \rfloor$, $\tilde{\omega}(\lfloor n\mathbf{u}_{1,d-1} \rfloor + i\mathbf{e}_{d-1}) = \omega(\mathbf{x}^i, i)$.
- (iii) Pick all other $\tilde{\omega}(\mathbf{y})$ independently of everything else.

Now, keeping π fixed, we project the paths down to \mathbb{Z}_+^{d-1} and create a partition function (marked by a tilde) in the new environment $\tilde{\omega}$.

$$\begin{aligned}
(3.17) \quad \log Z_\pi &= \log Z_{(\mathbf{0},0),(\mathbf{x}^1,0)} + \sum_{i=1}^{\lfloor nu_d \rfloor} \log Z_{(\mathbf{x}^i,i),(\mathbf{x}^{i+1},i)}^\square \\
&= \sum_{i=0}^{\lfloor nu_d \rfloor} \log Z_{(\mathbf{x}^i,i),(\mathbf{x}^{i+1},i)} + \sum_{i=1}^{\lfloor nu_d \rfloor} \omega(\mathbf{x}^i,i) \\
&= \sum_{i=0}^{\lfloor nu_d \rfloor} \log \tilde{Z}_{\mathbf{x}^i,\mathbf{x}^{i+1}} + \sum_{i=1}^{\lfloor nu_d \rfloor} \tilde{\omega}(\lfloor n\mathbf{u}_{1,d-1} \rfloor + i\mathbf{e}_{d-1}) \\
&\leq \log \tilde{Z}_{\lfloor n\mathbf{u}_{1,d-1} \rfloor + \lfloor nu_d \mathbf{e}_{d-1} \rfloor}.
\end{aligned}$$

Introduce the continuous functions ($1 \leq i < d$)

$$(3.18) \quad F_i(\mathbf{u}) = \sum_{j=1}^{i-1} ((u_j + u_i) \log(u_j + u_i) - u_j \log u_j - u_i \log u_i).$$

Counting the number of ways to decompose the length from 0 to $\lfloor nu_i \rfloor$ into $\lfloor nu_d \rfloor + 1$ segments and Stirling's formula give

$$\begin{aligned}
(3.19) \quad m_0 = |\Lambda_{\lfloor n\mathbf{u} \rfloor}| &= \prod_{1 \leq i \leq d-1} \binom{\lfloor nu_i \rfloor + \lfloor nu_d \rfloor}{\lfloor nu_d \rfloor + 1} = \exp\{nF_d(\mathbf{u}) + o(n)\} \\
&\leq \exp\{nF_d(\mathbf{u}) + n\eta\}
\end{aligned}$$

where the last inequality is valid for large n and we introduced a small $\eta > 0$ that we can send to zero after limits in n have been taken. By a union bound and the coupling (3.17) separately for each $\pi \in \Lambda_{\lfloor n\mathbf{u} \rfloor}$,

$$\begin{aligned}
-J_{\mathbf{u}}(r) &\leq \overline{\lim}_{n \rightarrow \infty} n^{-1} \log \sum_{\pi \in \Lambda_{\lfloor n\mathbf{u} \rfloor}} \mathbb{P}\{\log Z_\pi \geq nr - \log m_0\} \\
&\leq \lim_{n \rightarrow \infty} \left(\frac{\log m_0}{n} + n^{-1} \log \mathbb{P}\left\{ \log \tilde{Z}_{\lfloor n\mathbf{u}_{1,d-1} \rfloor + \lfloor nu_d \mathbf{e}_{d-1} \rfloor} \geq nr - nF_d(\mathbf{u}) - n\eta \right\} \right) \\
&= F_d(\mathbf{u}) - J_{\mathbf{u}_{1,d-1} + u_d \mathbf{e}_{d-1}}(r - F_d(\mathbf{u}) - \eta).
\end{aligned}$$

In the last step above a little correction as in (3.12) replaces $\lfloor n\mathbf{u}_{1,d-1} \rfloor + \lfloor nu_d \mathbf{e}_{d-1} \rfloor$ with $\lfloor n\mathbf{u}_{1,d-1} + nu_d \mathbf{e}_{d-1} \rfloor$.

Let $\tilde{\mathbf{u}}_{1,d} = \mathbf{u}$ and for $1 \leq i < d$,

$$\tilde{\mathbf{u}}_{1,i} = \mathbf{u}_{1,i} + \sum_{j=i+1}^d u_j \mathbf{e}_i \in \mathbb{Z}_+^i.$$

Proceeding inductively, we get the lower bound

$$(3.20) \quad J_{\mathbf{u}}(r) \geq J_{\tilde{\mathbf{u}}_{1,k}} \left(r - \sum_{k+1 \leq i \leq d} (F_i(\mathbf{u}) - \eta) \right) - \sum_{k+1 \leq i \leq d} F_i(\mathbf{u}).$$

On the right-hand side we have a rate function $J_{\tilde{\mathbf{u}}_{1,k}}$ with $\tilde{\mathbf{u}}_{1,k} \rightarrow \mathbf{v}_{1,k}$ in the interior of \mathbb{R}_+^k . Thus we have continuity. We can first let $\eta \searrow 0$. Then let $(\mathbf{u}, r) \rightarrow (\mathbf{v}, s)$. Note that $u_i \rightarrow 0$ implies $F_i(\mathbf{u}) \rightarrow 0$. Together all this gives the lower semicontinuity:

$$\lim_{(\mathbf{u}, r) \rightarrow (\mathbf{v}, s)} J_{\mathbf{u}}(r) \geq J_{\tilde{\mathbf{v}}_{1,k}}(s) = J_{\mathbf{v}}(s).$$

Now we know J is continuous on all of $(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \times \mathbb{R}$.

Let us observe limit (3.7). From one side we have

$$\mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor}^\square \geq nr\} \geq \mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor} \geq nr\} \mathbb{P}\{\omega(\mathbf{0}) \geq 0\}.$$

From the other, pick a coordinate $u_i > 0$, and for each n an integer $n < m_n < n + o(n)$ such that $2\mathbf{e}_i + \lfloor n\mathbf{u} \rfloor \leq \lfloor m_n\mathbf{u} \rfloor$. For each n fix a directed path $\{x_j^n\}$ from $2\mathbf{e}_i + \lfloor n\mathbf{u} \rfloor$ to $\lfloor m_n\mathbf{u} \rfloor$. Inequality

$$\omega(\mathbf{e}_i) + \log Z_{2\mathbf{e}_i, 2\mathbf{e}_i + \lfloor n\mathbf{u} \rfloor}^\square + \sum_j \omega(x_j^n) \leq \log Z_{\lfloor m_n\mathbf{u} \rfloor}$$

gives

$$\mathbb{P}\{\log Z_{\lfloor n\mathbf{u} \rfloor}^\square \geq nr\} \mathbb{P}\left\{\omega(\mathbf{e}_i) + \sum_j \omega(x_j^n) \geq 0\right\} \leq \mathbb{P}\{\log Z_{\lfloor m_n\mathbf{u} \rfloor} \geq nr\}$$

Assumption (3.5) and the continuity of J give the conclusion. \square

Proof of Theorem 3.3. It remains to prove continuity at $(\mathbf{0}, s)$. Let $(\mathbf{u}, r) \rightarrow (\mathbf{0}, s)$. Define the right-tail Cramér rate function for $a > 0$, $x \in \mathbb{R}$:

$$\kappa_a(x) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\left\{n^{-1} \sum_{i=1}^{\lfloor na \rfloor} \omega(x_i) \geq nx\right\} = \begin{cases} aI(x/a), & x \geq a\mathbb{E}[\omega(\mathbf{0})] \\ 0, & x \leq a\mathbb{E}[\omega(\mathbf{0})]. \end{cases}$$

Check that as $(a, x) \rightarrow (0, s)$, $\kappa_a(x) \rightarrow J_{\mathbf{0}}(s)$ defined by (3.9).

For upper semicontinuity, bound $Z_{\lfloor n\mathbf{u} \rfloor}$ below by a single path:

$$J_{\mathbf{u}}(r) \leq - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\left\{n^{-1} \sum_{i=1}^{\lfloor n\mathbf{u} \rfloor_1} \omega(x_i) \geq nr\right\} = \kappa_{\lfloor \mathbf{u} \rfloor_1}(r).$$

For lower semicontinuity, permute the coordinates so that $u_1 > 0$ as $\mathbf{u} \rightarrow \mathbf{0}$. Apply (3.20) after η has been taken to zero:

$$J_{\mathbf{u}}(r) \geq J_{u_1 \mathbf{e}_1} \left(r - \sum_{2 \leq i \leq d} F_i(\mathbf{u}) \right) - \sum_{2 \leq i \leq d} F_i(\mathbf{u}).$$

Since $J_{u_1 \mathbf{e}_1} = \kappa_{u_1}$ we get the lower semicontinuity. \square

Finally two lemmas for later use. The next one allows more general lattice sequences for the right-tail LDP.

LEMMA 3.5. *Let $\mathbf{y} \in (0, \infty)^d$ and $\mathbf{u}_n \in \mathbb{Z}_+^d$ be a sequence such that $n^{-1}\mathbf{u}_n \rightarrow \mathbf{y}$. Then for $r \in \mathbb{R}$,*

$$(3.21) \quad \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\mathbf{u}_n} \geq nr\} = -J_{\mathbf{y}}(r).$$

Proof. Let us use assumption (3.5) again. Since the coordinates of \mathbf{u}_n and $\lfloor n\mathbf{y} \rfloor$ are increasing to ∞ , for each n we can find ℓ_n and m_n such that $\lfloor \ell_n \mathbf{y} \rfloor \leq \mathbf{u}_n \leq \lfloor m_n \mathbf{y} \rfloor$ and in such a way that $n - \ell_n$, $n - m_n$ are eventually $o(n)$. For each n fix directed paths $\{x_{n,i}\}_{0 \leq i \leq K_n}$ from $\lfloor \ell_n \mathbf{y} \rfloor$ to \mathbf{u}_n and $\{x'_{n,j}\}_{0 \leq j \leq K'_n}$ from \mathbf{u}_n to $\lfloor m_n \mathbf{y} \rfloor$. Then

$$Z_{\lfloor \ell_n \mathbf{y} \rfloor} \cdot W_n \leq Z_{\mathbf{u}_n} \leq Z_{\lfloor m_n \mathbf{y} \rfloor} \cdot (W'_n)^{-1}$$

where

$$\log W_n = \sum_{1 \leq i \leq K_n} \omega(x_i) \quad \text{and} \quad \log W'_n = \sum_{1 \leq i \leq K'_n} \omega(x'_i).$$

Assumption $n^{-1}\mathbf{u}_n \rightarrow \mathbf{y}$ implies that K_n and K'_n are also $o(n)$.

The estimates we need follow. For example,

$$\mathbb{P}\{\log Z_{\lfloor m_n \mathbf{y} \rfloor} \geq nr\} \geq \mathbb{P}\{\log W'_n \geq 0\} \mathbb{P}\{\log Z_{\mathbf{u}_n} \geq nr\}$$

and then by assumption (3.5) and the continuity of the rate function,

$$\overline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\mathbf{u}_n} \geq nr\} \leq \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor m_n \mathbf{y} \rfloor} \geq nr\} = -J_{\mathbf{y}}(r).$$

Similarly for the complementary lower bound on $\underline{\lim}$. □

LEMMA 3.6. *Suppose that for each n , L_n and Z_n are independent random variables. Assume that the limits*

$$(3.22) \quad \lambda(s) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{L_n \geq ns\},$$

$$(3.23) \quad \phi(s) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{Z_n \geq ns\}$$

exist and are finite for all $s \in \mathbb{R}$. Assume that $\lambda(a_\lambda) = \phi(a_\phi) = 0$ for some $a_\lambda, a_\phi \in \mathbb{R}$. Assume also that λ is continuous. Then for $r \in \mathbb{R}$

$$(3.24) \quad \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{L_n + Z_n \geq nr\} = \begin{cases} - \inf_{a_\lambda \leq s \leq r - a_\phi} \{\phi(r - s) + \lambda(s)\}, & r > a_\phi + a_\lambda \\ 0, & r \leq a_\phi + a_\lambda. \end{cases}$$

Proof. The lower bound \geq follows from

$$\mathbb{P}\{L_n + Z_n \geq nr\} \geq \mathbb{P}\{L_n \geq ns\} \mathbb{P}\{Z_n \geq n(r - s)\}.$$

Since an upper bound 0 is obvious, it remains to show the upper bound for the case $r > a_\phi + a_\lambda$. Take a finite partition $a_\lambda = q_0 < \dots < q_m = r - a_\phi$. Then use a union bound and independence:

$$\begin{aligned} & \mathbb{P}\{L_n + Z_n \geq nr\} \\ & \leq \mathbb{P}\{L_n + Z_n \geq nr, L_n < nq_0\} \\ & \quad + \sum_{i=0}^{m-1} \mathbb{P}\{L_n + Z_n \geq nr, nq_i \leq L_n \leq nq_{i+1}\} + \mathbb{P}\{L_n \geq nq_m\} \\ & \leq \mathbb{P}\{Z_n \geq n(r - q_0)\} + \sum_{i=0}^{m-1} \mathbb{P}\{Z_n \geq n(r - q_{i+1})\} \mathbb{P}\{L_n \geq nq_i\} + \mathbb{P}\{L_n \geq nq_m\}. \end{aligned}$$

From this

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{L_n + Z_n \geq nr\} \\ & \leq -\min \left\{ \phi(r - q_0), \min_{0 \leq i \leq m-1} [\phi(r - q_{i+1}) + \lambda(q_i)], \lambda(q_m) \right\}. \end{aligned}$$

Note that $\lambda(q_0) = \phi(r - q_m) = 0$, refine the partition and use the continuity of λ . \square

4. PROOFS FOR THE I.I.D. LOG-GAMMA MODEL

In this section we prove the results of Section 2.1. Throughout this section the dimension $d = 2$ and the weights satisfy $Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$ as in (2.1). As before, for $(s, t) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$ define the function $J_{s,t}$ by the limit

$$(4.1) \quad J_{s,t}(r) = -\lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{[ns],[nt]} \geq nr\}, \quad r \in \mathbb{R}.$$

At the origin set

$$(4.2) \quad J_{0,0}(r) = \begin{cases} 0, & r \leq 0, \\ \mu r, & r \geq 0. \end{cases}$$

Then, as observed in Remark 3.4, the function $J_{s,t}(r)$ is finite and continuous at all $(s, t, r) \in \mathbb{R}_+^2 \times \mathbb{R}$.

We begin with a lemma that proves Theorem 2.1.

LEMMA 4.1. *For $(s, t) \in \mathbb{R}_+^2$ the limiting free energy of (2.5) satisfies*

$$(4.3) \quad p(s, t) = \inf_{0 < \theta < \mu} \{-s\Psi_0(\theta) - t\Psi_0(\mu - \theta)\}.$$

The infimum is achieved at some θ because $\Psi_0(0+) = -\infty$.

Proof. The proof anticipates some themes of the later LDP proof but in a simpler context. We already recorded the law of large numbers (2.18). The decomposition (see Figure 2)

$$(4.4) \quad Z_{[ns],[nt]}^{(\theta)} = \sum_{k=1}^{[ns]} \left(\prod_{i=1}^k U_{i,0} \right) Z_{(k,1),([ns],[nt])}^{\square} + \sum_{\ell=1}^{[nt]} \left(\prod_{j=1}^{\ell} V_{0,j} \right) Z_{(1,\ell),([ns],[nt])}^{\square}.$$

from (2.14) gives asymptotically

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-1} \log Z_{[ns],[nt]}^{(\theta)} &= \lim_{n \rightarrow \infty} \left\{ \max_{1 \leq k \leq [ns]} \left(n^{-1} \sum_{i=1}^k \log U_{i,0} + n^{-1} \log Z_{(k,1),([ns],[nt])}^{\square} \right) \right. \\ &\quad \left. \bigvee \max_{1 \leq \ell \leq [nt]} \left(n^{-1} \sum_{j=1}^{\ell} \log V_{0,j} + n^{-1} \log Z_{(1,\ell),([ns],[nt])}^{\square} \right) \right\}. \end{aligned}$$

This can be coarse-grained with readily controllable errors of sums of independent variables. We omit the details since similar arguments appear elsewhere in the paper. The conclusion is the alternative formula

$$(4.5) \quad \begin{aligned} & \lim_{n \rightarrow \infty} n^{-1} \log Z_{[ns],[nt]}^{(\theta)} \\ &= \sup_{0 \leq a \leq s} \{-a\Psi_0(\theta) + p(s - a, t)\} \bigvee \sup_{0 \leq b \leq t} \{-b\Psi_0(\mu - \theta) + p(s, t - b)\}. \end{aligned}$$

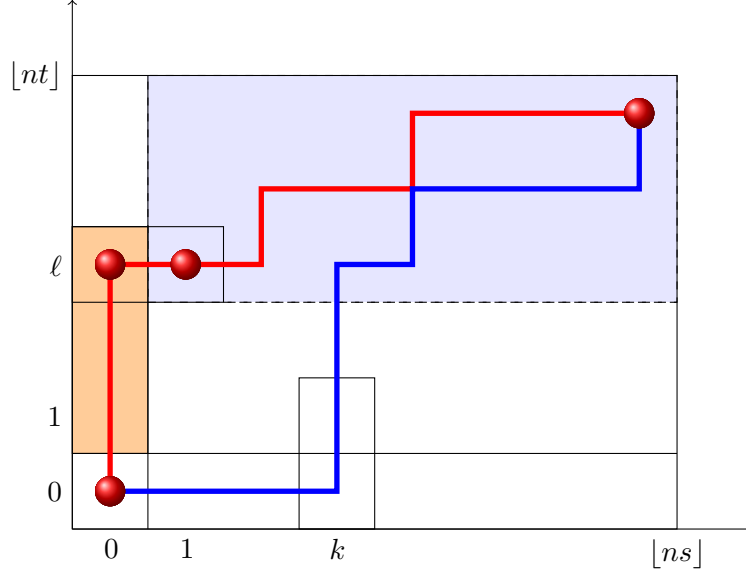


FIGURE 2. Graphical representation of the decomposition in equation (4.4).

Take $s = t$, combine (2.18) and (4.5), and use the symmetry $p(s, t) = p(t, s)$ to get

$$-t(\Psi_0(\theta) + \Psi_0(\mu - \theta)) = \sup_{0 \leq a \leq t} \{ -a(\Psi_0(\theta) \wedge \Psi_0(\mu - \theta)) + p(t - a, t) \}.$$

Take $\theta \in (0, \mu/2]$ so that $\Psi_0(\theta) \leq \Psi_0(\mu - \theta)$ [Ψ_0 is strictly increasing] and set $a = t - s$:

$$-t\Psi_0(\mu - \theta) = \sup_{0 \leq s \leq t} \{ s\Psi_0(\theta) + p(s, t) \}.$$

Turn this into a convex duality through the change of variable $v = \Psi_0(\theta)$:

$$(4.6) \quad -t\Psi_0(\mu - \Psi_0^{-1}(v)) = \sup_{0 \leq s \leq t} \{ sv + p(s, t) \}, \quad v \in (-\infty, \Psi_0(\mu/2)].$$

It follows from the limit definition of $p(s, t)$ that it is concave and continuous in $s \in [0, t]$. Extend $f(s) = -p(s, t)$ to a lower semicontinuous convex function of $s \in \mathbb{R}$ by setting $f(s) = \infty$ for $s \notin [0, t]$. Then (4.6) tells us that

$$f^*(v) = -t\Psi_0(\mu - \Psi_0^{-1}(v)) \quad \text{for } v \in (-\infty, \Psi_0(\mu/2)].$$

From this we can differentiate to get $\lim_{v \searrow -\infty} (f^*)'(v) = 0$ and $(f^*)'(\Psi_0(\mu/2)) = t$. These derivative values imply that for $s \in [0, t]$ the supremum in the double convex duality can be restricted as follows:

$$f(s) = \sup_{v \in (-\infty, \Psi_0(\mu/2)]} \{ vs - f^*(v) \}.$$

Undoing the change of variables turns this equation into (4.3) which is thereby proved. \square

The next lemma gives left tail bounds strong enough to imply $I_{s,t}(r) = \infty$ for $r < p(s, t)$, and the same result for the stationary model. The proof is a straightforward coarse-graining argument. We do not expect the results to be optimal.

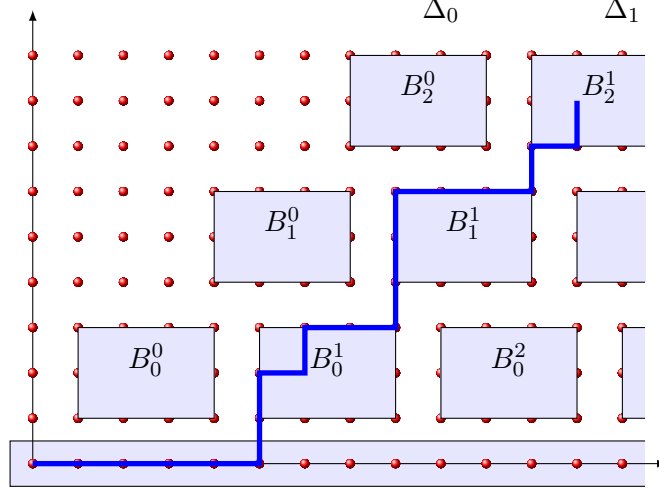


FIGURE 3. The $\lfloor ms \rfloor \times \lfloor mt \rfloor$ rectangles and the diagonals Δ_i in the proof of Lemma 4.2. The thickset line is a lattice path that is counted in Z_1 .

LEMMA 4.2. *Fix $0 < a < 1$. Then there exist constants $0 < c, C < \infty$ that depend on the parameters given below, so that the following estimates hold.*

(a) *For $(s, t) \in (0, \infty)^2$ and $r < p(s, t)$,*

$$(4.7) \quad \mathbb{P}\{\log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} \leq nr\} \leq Ce^{-cn^{1+a}} \quad \text{for all } n \geq 1.$$

(b) *For $(s, t) = \alpha(\Psi_1(\mu - \theta), \Psi_1(\theta))$ for some $\alpha > 0$, parallel to the characteristic direction, and $r < p^{(\theta)}(s, t)$,*

$$(4.8) \quad \mathbb{P}\{\log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor}^{(\theta)} \leq nr\} \leq Ce^{-cn^{1+a}} \quad \text{for all } n \geq 1.$$

Proof. We give a proof of (b) with some details left sketchy. Part (a) has a similar proof. We bound $Z_{\lfloor ns \rfloor, \lfloor nt \rfloor}^{(\theta)}$ from below by considering a subset of lattice paths, arranged in a collection of i.i.d. partition functions over subsets of the rectangle.

The choice of (s, t) implies that $p^{(\theta)}(s, t) = p(s, t)$. Fix $0 < \varepsilon < (p^{(\theta)}(s, t) - r)/4$. Fix $m \in \mathbb{N}$ large enough so that $m(s \wedge t) \geq 1$ and

$$(4.9) \quad \mathbb{E} \log Z_{\lfloor ms \rfloor, \lfloor mt \rfloor} > m(r + 2\varepsilon).$$

Let $B_{a,b}^{k,\ell} = \{a, \dots, a + k - 1\} \times \{b, \dots, \ell + b - 1\}$ denote the $k \times \ell$ rectangle with lower left corner at (a, b) . For $i, \ell \geq 0$ define pairwise disjoint $\lfloor ms \rfloor \times \lfloor mt \rfloor$ rectangles

$$B_\ell^i = B_{(\ell+i)\lfloor ms \rfloor - \ell + 1, \ell\lfloor mt \rfloor + 1}^{\lfloor ms \rfloor, \lfloor mt \rfloor}.$$

Define a diagonal union of these rectangles by $\Delta_i = \bigcup_{\ell \geq 0} B_\ell^i$, $i \geq 0$. (See Figure 3).

Let $M = \lfloor n^a \rfloor \lfloor ms \rfloor$. This is the range of diagonals Δ_i we consider. Then we cut the diagonals off before they exit the $\lfloor ns \rfloor \times \lfloor nt \rfloor$ rectangle. Let $N = N(n)$ be the maximal

integer such that B_N^M lies in $[0, \lfloor ns \rfloor] \times [0, \lfloor nt \rfloor]$. Diagonal Δ_M exits the $\lfloor ns \rfloor \times \lfloor nt \rfloor$ rectangle through the east edge, and consequently there exist positive constants c_m, C_m such that

$$(4.10) \quad \lfloor ns \rfloor - c_m < N \lfloor ms \rfloor + \lfloor ms \rfloor \lfloor n^a \rfloor \leq \lfloor ns \rfloor, \quad \text{and} \quad \lfloor nt \rfloor - C_m n^a < N \lfloor mt \rfloor \leq \lfloor nt \rfloor.$$

Having defined the cutoff N , define the other diagonals by $\Delta_\ell^n = \bigcup_{0 \leq \ell \leq N} B_\ell^i$ for $0 \leq i \leq M$. These diagonals lie in $[0, \lfloor ns \rfloor] \times [0, \lfloor nt \rfloor]$. Fix a path π that proceeds horizontally from $(N \lfloor ms \rfloor, N \lfloor mt \rfloor + 1)$ to $(\lfloor ns \rfloor, N \lfloor mt \rfloor + 1)$ and then vertically up to $(\lfloor ns \rfloor, \lfloor nt \rfloor)$. The number of lattice points on π is a constant multiple of n^a .

For $0 \leq i \leq M$ let Z_i denote the partition function of paths x_\cdot of the following type: x_\cdot proceeds along the x -axis from the origin to $(i \lfloor ms \rfloor + 1, 0)$, enters Δ_i^n at $(i \lfloor ms \rfloor + 1, 1)$, and stays in Δ_i^n until it exits from the upper right corner of B_N^i with a vertical step that connects it with π . After that x_\cdot follows π to $(\lfloor ns \rfloor, \lfloor nt \rfloor)$. The number K of points on x_\cdot outside Δ_i^n is independent of i and bounded by a constant multiple of n^a . Let

$$X = \min\{Y_x : x \in \pi \text{ or } x \in \{(i, 0) : 0 \leq i \leq M\}\}$$

be the minimal weight outside Δ_i^n encountered by any path x_\cdot of Z_i , for any $0 \leq i \leq M$.

Let Z_i^Δ be the partition function of all lattice paths in Δ_i^n from the lower left corner of B_0^i to the upper right corner of B_N^i . Then $Z_i \geq X^K Z_i^\Delta$, and consequently

$$\begin{aligned} \mathbb{P}\{\log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor}^{(\theta)} \leq nr\} &\leq \mathbb{P}\left\{\log \sum_{i=0}^M X^K Z_i^\Delta \leq nr\right\} \\ &= \mathbb{P}\left\{K \log X + \log \sum_{i=0}^M Z_i^\Delta \leq nr\right\} \\ (4.11) \quad &\leq \mathbb{P}\{K \log X \leq -n\varepsilon\} + \mathbb{P}\left\{\log \sum_{i=0}^M Z_i^\Delta \leq n(r + \varepsilon)\right\}. \end{aligned}$$

Explicit computation with the gamma distribution and $K \leq cn^a$ give $\mathbb{P}\{K \log X \leq -n\varepsilon\} \leq e^{-n^2}$ for large n .

The $\{Z_i^\Delta\}$ are i.i.d., and Z_0^Δ is a product of the i.i.d. partition functions Z_k^0 of the individual rectangles B_k^0 whose mean was controlled by (4.9). A standard large deviation estimate for an i.i.d. sum gives

$$\begin{aligned} \mathbb{P}\left\{\log \sum_{i=0}^M Z_i^\Delta \leq n(r + \varepsilon)\right\} &\leq \mathbb{P}\{\log Z_0^\Delta \leq n(r + \varepsilon)\}^M = \mathbb{P}\left\{\sum_{k=0}^N \log Z_k^0 \leq n(r + \varepsilon)\right\}^M \\ &= \mathbb{P}\left\{\sum_{k=0}^{n/m+o(n)} \log Z_k^0 \leq n(r + \varepsilon)\right\}^M \leq e^{-cnM} \leq e^{-c_1 n^{1+a}}. \end{aligned}$$

Putting these bounds back on line (4.11) completes the proof of (4.8). \square

The main work resides in proving the following right tail result.

PROPOSITION 4.3. *Let $(s, t) \in \mathbb{R}^2$. Then for all $r \in \mathbb{R}$, $J_{s,t}(r)$ is given by*

$$(4.12) \quad J_{s,t}(r) = \sup_{\xi \in [0, \mu)} \left\{ r\xi - \inf_{\theta \in (\xi, \mu)} (tM_\theta(\xi) - sM_{\mu-\theta}(-\xi)) \right\}.$$

Before turning to the proof of Proposition 4.3 let us observe how Theorem 2.2 follows.

Proof of Theorem 2.2. Only a few simple observations are required. Start by defining $I_{s,t}$ as given in (2.6). Then formula (2.9) that connects $I_{s,t}$ and $J_{s,t}$ is established by (4.12) and by knowing that $J_{s,t}(r) = 0$ for $r \leq p(s,t)$ (Theorem 3.2). The regularity properties of $I_{s,t}$ follow from the general properties of J in Theorems 3.2 and 3.3.

The upper large deviation bound (2.8) is built into (4.7) and (4.1).

For the lower large deviation bound (2.7) we consider three cases.

(i) If $p(s,t) \in G$ then $\mathbb{P}\{n^{-1} \log Z_{[ns],[nt]} \in G\} \rightarrow 1$ and (2.7) holds trivially because its right-hand side is ≤ 0 .

(ii) If $G \subseteq (-\infty, p(s,t))$, (2.7) holds trivially because its right-hand side is $-\infty$.

(iii) The remaining case is the one where G contains an interval $(a,b) \subset (p(s,t), \infty)$. Since the distribution is continuous including a into G makes no difference, and so

$$\begin{aligned} & n^{-1} \log \mathbb{P}\{n^{-1} \log Z_{[ns],[nt]} \in G\} \\ & \geq n^{-1} \log (\mathbb{P}\{\log Z_{[ns],[nt]} \geq na\} - \mathbb{P}\{\log Z_{[ns],[nt]} \geq nb\}) \\ & \rightarrow -J_{s,t}(a) \end{aligned}$$

where the limit follows from (4.1) and the strict increasingness of $J_{s,t}$ on $[p(s,t), \infty)$ which implies that for large enough n

$$\mathbb{P}\{\log Z_{[ns],[nt]} \geq nb\} \leq e^{-nJ_{s,t}(a) - n\varepsilon}$$

for some $\varepsilon > 0$. We can take $a = \inf G \cap (p(s,t), \infty)$ and then $J_{s,t}(a) = \inf_{r \in G \cap (p(s,t), \infty)} I_{s,t}(r) = \inf_{r \in G} I_{s,t}(r)$. \square

The remainder of the section is devoted to proving Proposition 4.3. Again we begin with the decomposition (4.4) of the stationary partition function. Inside the sums on the right of (4.4) we have partition functions with i.i.d. Gamma $^{-1}(\mu)$ -weights $\{Y_{i,j}\}$ whose large deviations we wish to extract. But we do not know the large deviations of $\log Z_{[ns],[nt]}^{(\theta)}$ so at first the decomposition seems unhelpful. To get around the problem, use definition (2.15) to write

$$\log Z_{[ns],[nt]}^{(\theta)} - \log Z_{0,[nt]}^{(\theta)} = \sum_{j=1}^{[ns]} \log U_{i,[nt]}.$$

By Proposition 2.10 we have a sum of i.i.d.'s on the right, whose large deviations we can immediately write down by Cramér's theorem. To take advantage of this, divide through (4.4) by $Z_{0,[nt]}^{(\theta)} = \prod_{j=1}^{[nt]} V_{0,j}$ to rewrite it as

$$\begin{aligned} (4.13) \quad \prod_{i=1}^{[ns]} U_{i,[nt]} &= \sum_{\ell=1}^{[nt]} \left(\prod_{j=\ell+1}^{[nt]} V_{0,j}^{-1} \right) Z_{(1,\ell),([ns],[nt])}^{\square} \\ &+ \sum_{k=1}^{[ns]} \left(\prod_{j=1}^{[nt]} V_{0,j}^{-1} \right) \left(\prod_{i=1}^k U_{i,0} \right) Z_{(k,1),([ns],[nt])}^{\square}. \end{aligned}$$

To compactify notation we use a convention where the y -axis is labeled by negative indices and introduce these quantities:

$$(4.14) \quad \text{for } k \in \mathbb{Z}, \quad \eta_k = \begin{cases} \prod_{j=-k+1}^{\lfloor nt \rfloor} V_{0,j}^{-1}, & k \leq 0, \\ \left(\prod_{j=1}^{\lfloor nt \rfloor} V_{0,j}^{-1} \right) \prod_{i=1}^k U_{i,0}, & k \geq 1, \end{cases}$$

where an empty product equals 1 by definition, and

$$(4.15) \quad \text{for } z \in \mathbb{R}, \quad \mathbf{v}(z) = \begin{cases} (1, \lfloor -z \rfloor), & z \leq -1, \\ (1, 1), & -1 < z < 1, \\ (\lfloor z \rfloor, 1), & z \geq 1. \end{cases}$$

Then (4.13) rewrites as

$$(4.16) \quad \prod_{i=1}^{\lfloor ns \rfloor} U_{i, \lfloor nt \rfloor} = \sum_{\substack{k=-\lfloor nt \rfloor \\ k \neq 0}}^{\lfloor ns \rfloor} \eta_k Z_{\mathbf{v}(k), (\lfloor ns \rfloor, \lfloor nt \rfloor)}^{\square}$$

from which we extract these inequalities:

$$(4.17) \quad \begin{aligned} \log \eta_k + \log Z_{\mathbf{v}(k), (\lfloor ns \rfloor, \lfloor nt \rfloor)}^{\square} &\leq \sum_{i=1}^{\lfloor ns \rfloor} \log U_{i, \lfloor nt \rfloor} \\ &\leq \max_{\substack{-\lfloor nt \rfloor \leq k \leq \lfloor ns \rfloor \\ k \neq 0}} \{ \log \eta_k + \log Z_{\mathbf{v}(k), (\lfloor ns \rfloor, \lfloor nt \rfloor)}^{\square} \} + \log(n(s+t)). \end{aligned}$$

These inequalities will be the basis for proving Proposition 4.3.

We record the right tail rate functions for the random variables in (4.17).

For the i.i.d. weights $\{U_{i, \lfloor nt \rfloor}\}$ we have the right branch of the Cramér rate function

$$(4.18) \quad R_s(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \left\{ \sum_{i=1}^{\lfloor ns \rfloor} \log U_{i, \lfloor nt \rfloor} \geq nr \right\} = \begin{cases} s I_{\theta}(rs^{-1}), & r \geq -s\Psi_0(\theta) \\ 0, & r < -s\Psi_0(\theta). \end{cases}$$

The rate function I_{θ} defined by (4.18) is given by

$$(4.19) \quad I_{\theta}(r) = -r\Psi_0^{-1}(-r) - \log \Gamma(\Psi_0^{-1}(-r)) + \theta r + \log \Gamma(\theta), \quad r \in \mathbb{R}.$$

The convex dual of R_s is given by

$$(4.20) \quad R_s^*(\xi) = \begin{cases} s \log \Gamma(\theta - \xi) - s \log \Gamma(\theta), & 0 \leq \xi < \theta \\ \infty, & \xi < 0 \text{ or } \xi \geq \theta, \end{cases}$$

and we emphasize that it can be finite only when $\theta > \xi \geq 0$.

For real $a \in [-t, s]$

$$(4.21) \quad \kappa_a(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \{ \log \eta_{\lfloor na \rfloor} \geq nr \}$$

exists and is finite, convex and continuous in r . (For $a \leq 0$ it is simply a Cramér rate function for an i.i.d. sum, and for $a > 0$ we can use Lemma 3.6.) The convex dual is

$$(4.22) \quad \begin{aligned} \kappa_a^*(\xi) &= \sup_{r \in \mathbb{R}} \{\xi r - \kappa_a(r)\} \\ &= \begin{cases} (t+a)(\log \Gamma(\mu - \theta + \xi) - \log \Gamma(\mu - \theta)), & -t \leq a \leq 0, \xi \geq 0, \\ t(\log \Gamma(\mu - \theta + \xi) - \log \Gamma(\mu - \theta)) + a(\log \Gamma(\theta - \xi) - \log \Gamma(\theta)) & 0 < a \leq s, 0 \leq \xi < \theta, \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

The derivation of (4.22) is similar to that of (4.20) from (4.18). Note that there is a discontinuity in κ_a and κ_a^* as a passes through 0. The rightmost zero $m_{\kappa,a}$ of κ_a is the law of large numbers limit:

$$(4.23) \quad m_{\kappa,a} = \lim_{n \rightarrow \infty} \frac{\log \eta_{[na]}}{n} = \begin{cases} (t+a)\Psi_0(\mu - \theta), & -t \leq a \leq 0 \\ t\Psi_0(\mu - \theta) - a\Psi_0(\theta), & 0 < a \leq s. \end{cases}$$

In contrast to the functions κ_a and κ_a^* , $m_{\kappa,a}$ is continuous at $a = 0$. Introduce the “macroscopic” version of (4.15): for real a ,

$$(4.24) \quad n^{-1}\mathbf{v}(na) \rightarrow \bar{\mathbf{v}}(a) = \begin{cases} (0, -a), & -t \leq a \leq 0, \\ (a, 0), & 0 \leq a \leq s. \end{cases}$$

With this notation we have, again for real $a \in [-t, s]$, for the partition functions that appear in (4.16) these large deviations:

$$(4.25) \quad J_{(s,t)-\bar{\mathbf{v}}(a)}(r) = - \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log Z_{\mathbf{v}(na), ([ns], [nt])}^\square \geq nr\}.$$

We used Lemma 3.5 to take care of the small discrepancy between $([ns], [nt]) - \mathbf{v}(na)$ and $[n((s, t) - \bar{\mathbf{v}}(a))]$, unless $a = -t$ or $a = s$ when this is a case of i.i.d. large deviations, and therefore simpler.

Let $m_{\kappa,a}$ and $m_{J,b}$ be the rightmost zeroes of κ_a and $J_{(s,t)-\bar{\mathbf{v}}(b)}$ respectively. For $(a, b) \in [-t, s]^2$ let

$$(4.26) \quad \begin{aligned} H_{s,t}^{a,b}(r) &= \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{\log \eta_{[na]} + \log Z_{\mathbf{v}(nb), ([ns], [nt])}^\square \geq nr\} \\ &= \begin{cases} 0, & r < m_{\kappa,a} + m_{J,b} \\ \inf_{m_{\kappa,a} \leq x \leq r - m_{J,b}} \{\kappa_a(x) + J_{(s,t)-\bar{\mathbf{v}}(b)}(r - x)\}, & r \geq m_{\kappa,a} + m_{J,b}. \end{cases} \end{aligned}$$

The existence of $H_{s,t}^{a,b}(r)$ and the second equality follow from Lemma 3.6. We need some regularity:

LEMMA 4.4. *Fix $0 < s, t < \infty$ and a compact set $K \subseteq \mathbb{R}$. Then $H_{s,t}^{a,b}(r)$ is uniformly continuous as a function of $(b, r) \in [-t, s] \times K$, uniformly in $a \in [-t, s]$. That is,*

$$(4.27) \quad \lim_{\delta \searrow 0} \sup_{\substack{a, b, b' \in [-t, s], r, x \in K: \\ |b-b'| \leq \delta, |r-x| \leq \delta}} |H_{s,t}^{a,b}(r) - H_{s,t}^{a,b'}(x)| = 0.$$

Proof. This follows from the explicit formula in (4.26). First, we have the joint continuity $(b, r) \mapsto J_{(s,t)-\mathbf{v}(b)}(r)$ from Theorem 3.3. Second, we argue that x in the infimum can be restricted to a single compact set simultaneously for $(a, b, r) \in [-t, s]^2 \times K$. That $m_{\kappa, a}$ is bounded is evident from (4.23). To show that the upper bound $r - m_{J, b}$ of x is bounded above, we need to show a lower bound on $m_{J, b} = p((s, t) - \mathbf{v}(b))$. A lower bound on the free energy is easy: by discarding all but a single path,

$$p((s, t) - \mathbf{v}(b)) = \lim_{n \rightarrow \infty} n^{-1} \log Z_{[n((s,t)-\mathbf{v}(b))]}^{\square} \geq -(s + t - |b|)\Psi_0(\mu). \quad \square$$

We abbreviate $H_{s,t}^a(r) = H_{s,t}^{a,a}(r)$.

The unknown rate functions $J_{s,t}$ are now inside (4.26), while the other rates R_s and κ_a we know explicitly. The next lemma is the counterpart of (4.17) in terms of rate functions.

LEMMA 4.5. *Let $s, t > 0$ and $r \in \mathbb{R}$. Then*

$$(4.28) \quad R_s(r) = \inf_{-t \leq a \leq s} H_{s,t}^a(r).$$

Proof. For any $a \in [-t, s]$, by the first inequality of (4.17),

$$\begin{aligned} -R_s(r) &= \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \left\{ \sum_{i=1}^{\lfloor ns \rfloor} \log U_{i, \lfloor nt \rfloor} \geq nr \right\} \\ &\geq \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P} \left\{ \log \eta_{\lfloor na \rfloor} + \log Z_{\mathbf{v}(na), (\lfloor ns \rfloor, \lfloor nt \rfloor)}^{\square} \geq nr \right\} \\ (4.29) \quad &\geq -H_{s,t}^a(r). \end{aligned}$$

Supremum over $a \in [-t, s]$ on the right gives \leq in (4.28).

To get \geq in (4.28) we use the second inequality of (4.17) together with a partitioning argument. Let $\varepsilon > 0$. Note this technical point about handling the errors of the partitioning. With $B, \delta > 0$, Chebyshev's inequality and the l.m.g.f. of (2.2) give the bound

$$(4.30) \quad \mathbb{P} \left\{ \sum_{i=1}^{\lfloor n\delta \rfloor} \log Y_{i,1} \leq -n\varepsilon \right\} \leq e^{-nB(\varepsilon - B^{-1}\delta \log \frac{\Gamma(\mu+B)}{\Gamma(\mu)})} \leq e^{-B\varepsilon n/2}$$

where the second inequality comes from choosing $\delta = \delta(\varepsilon, B)$ small enough. The right tail for $\log Y$ does not give such a bound with an arbitrarily large B . Consequently we arrange the errors so that they can be bounded as above.

Given $B > 0$, fix a small enough $\delta > 0$ and let $-t = a_0 < a_1 < \dots < a_q = 0 < \dots < a_m = s$ be a partition of the interval $[-t, s]$ so that $|a_{i+1} - a_i| < \delta$. We illustrate how a term with index k from the right-hand side of (4.17) is reduced to a term involving only

partition points. Consider the case $a_i \geq 0$ and let $\lfloor na_i \rfloor \leq k \leq \lfloor na_{i+1} \rfloor$.

$$\begin{aligned}
& \mathbb{P}\{\log \eta_k + \log Z_{\mathbf{v}(k),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq nr\} \\
& \leq \mathbb{P}\left\{\log \eta_{\lfloor na_{i+1} \rfloor} + \log Z_{\mathbf{v}(na_i),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square - \sum_{j=k+1}^{\lfloor na_{i+1} \rfloor} \log U_{j,0} - \sum_{j=\lfloor na_i \rfloor}^{k-1} \log Y_{j,1} \geq nr\right\} \\
& \leq \mathbb{P}\{\log \eta_{\lfloor na_{i+1} \rfloor} + \log Z_{\mathbf{v}(na_i),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq n(r - \varepsilon)\} \\
& \quad + \mathbb{P}\left\{- \sum_{j=k+1}^{\lfloor na_{i+1} \rfloor} \log U_{j,0} - \sum_{j=\lfloor na_i \rfloor}^{k-1} \log Y_{j,1} \geq n\varepsilon\right\} \\
(4.31) \quad & \leq \mathbb{P}\{\log \eta_{\lfloor na_{i+1} \rfloor} + \log Z_{\mathbf{v}(na_i),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq n(r - \varepsilon)\} + e^{-B\varepsilon n/2}.
\end{aligned}$$

On the other hand, if $a_i < 0$ and $\lfloor -na_{i+1} \rfloor < -k \leq \lfloor -na_i \rfloor$, then we would develop as follows:

$$\begin{aligned}
& \log \eta_k + \log Z_{\mathbf{v}(k),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \\
& \leq \log \eta_{\lfloor na_i \rfloor} - \sum_{j=-k+1}^{-\lfloor na_i \rfloor} \log V_{0,j} + \log Z_{\mathbf{v}(na_{i+1}),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square - \sum_{j=\lfloor -na_{i+1} \rfloor \vee 1}^{-k-1} \log Y_{1,j}
\end{aligned}$$

and get the same bound as on line (4.31) but with a_i and a_{i+1} switched around.

Now for \geq in (4.28). Assume n is large enough so that $n\varepsilon > \log(ns + nt)$. Starting from (4.17)

$$\begin{aligned}
& n^{-1} \log \mathbb{P}\left\{\sum_{i=1}^{\lfloor ns \rfloor} \log U_{i,\lfloor nt \rfloor} \geq nr\right\} \\
& \leq \max_{\substack{-\lfloor nt \rfloor \leq k \leq \lfloor ns \rfloor \\ k \neq 0}} n^{-1} \log \mathbb{P}\{\log \eta_k + \log Z_{\mathbf{v}(k),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq n(r - \varepsilon)\} + n^{-1} \log(ns + nt) \\
& \leq \max_{0 \leq i \leq q-1} n^{-1} \log \left(\mathbb{P}\{\log \eta_{\lfloor na_i \rfloor} + \log Z_{\mathbf{v}(na_{i+1}),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq n(r - 2\varepsilon)\} + e^{-B\varepsilon n/2} \right) \\
& \quad \vee \max_{q \leq i \leq m-1} n^{-1} \log \left(\mathbb{P}\{\log \eta_{\lfloor na_{i+1} \rfloor} + \log Z_{\mathbf{v}(na_i),(\lfloor ns \rfloor, \lfloor nt \rfloor)}^\square \geq n(r - 2\varepsilon)\} + e^{-B\varepsilon n/2} \right) + \varepsilon.
\end{aligned}$$

Take $n \rightarrow \infty$ above to obtain

$$\begin{aligned}
-R_s(r) & \leq \left\{ \max_{0 \leq i \leq q-1} (-H_{s,t}^{a_i, a_{i+1}}(r - 2\varepsilon)) \vee (-B\varepsilon/2) \right\} \\
& \quad \vee \left\{ \max_{q \leq i \leq m-1} (-H_{s,t}^{a_{i+1}, a_i}(r - 2\varepsilon)) \vee (-B\varepsilon/2) \right\} + \varepsilon \\
& \leq \sup_{a, b \in [-t, s]: |a-b| \leq \delta} (-H_{s,t}^{a,b}(r - 2\varepsilon)) \vee (-B\varepsilon/2) + \varepsilon.
\end{aligned}$$

We first let $\delta \searrow 0$, and by Lemma 4.4 the bound above becomes

$$-R_s(r) \leq \sup_{a \in [-t, s]} (-H_{s,t}^{a,a}(r - 2\varepsilon)) \vee (-B\varepsilon/2) + \varepsilon.$$

Next we take $B \nearrow \infty$, and finally $\varepsilon \searrow 0$ with another application of Lemma 4.4. This establishes \geq in (4.28). \square

A key analytic trick will be to look at the dual $J_{(t,t)-\bar{\nu}(a)}^*(\xi)$ of the right tail rate as a function of a . This lemma will be helpful.

LEMMA 4.6. *For a fixed $\xi \in [0, \mu)$ the function*

$$(4.32) \quad G_\xi(a) = \begin{cases} -J_{(t,t)-\bar{\nu}(a)}^*(\xi), & a \in [0, t] \\ \infty, & a < 0 \text{ or } a > t \end{cases}$$

*is continuous on $[0, t]$, and convex and lower semi-continuous on \mathbb{R} . In particular, $G_\xi^{**}(a) = G_\xi(a)$ for $a \in \mathbb{R}$.*

Proof. To show convexity on $[0, t]$ let $\lambda \in (0, 1)$ and $a = \lambda a_1 + (1 - \lambda)a_2$:

$$\begin{aligned} -J_{(t,t)-\bar{\nu}(a)}^*(\xi) &= -\sup_{r \in \mathbb{R}} \{\xi r - J_{(t,t)-\bar{\nu}(a)}(r)\} \\ &= \inf_{r \in \mathbb{R}} \{J_{t-a,t}(r) - \xi r\} \\ &\leq \inf_{r \in \mathbb{R}} \inf_{\substack{(r_1, r_2): \\ \lambda r_1 + (1-\lambda)r_2 = r}} \{\lambda(J_{t-a_1,t}(r_1) - \xi r_1) + (1-\lambda)(J_{t-a_2,t}(r_2) - \xi r_2)\} \\ &= \inf_{(r_1, r_2) \in \mathbb{R}^2} \{\lambda(J_{t-a_1,t}(r_1) - \xi r_1) + (1-\lambda)(J_{t-a_2,t}(r_2) - \xi r_2)\} \\ &= \lambda \inf_{r_1 \in \mathbb{R}} \{J_{t-a_1,t}(r_1) - \xi r_1\} + (1-\lambda) \inf_{r_2 \in \mathbb{R}} \{J_{t-a_2,t}(r_2) - \xi r_2\} \\ (4.33) \quad &= -\lambda J_{t-a_1,t}^*(\xi) - (1-\lambda) J_{t-a_2,t}^*(\xi). \end{aligned}$$

The inequality comes from the convexity of J in the variable $(t-a, t, r)$.

For finiteness on $[0, t]$ it is now enough to show that $G_\xi(a)$ is finite at the endpoints. Continuity then follows in the interior $(0, t)$. First take $a = t$. Then $J_{0,t}^*$ is the dual of a Cramér rate function, and for $\xi \geq 0$

$$(4.34) \quad G_\xi(t) = -J_{0,t}^*(\xi) = -t \log \mathbb{E} e^{\xi \log Y_{1,0}}$$

which is finite for $\xi < \mu$.

Convexity of $J_{s,t}(r)$ and symmetry $J_{s,t}(r) = J_{t,s}(r)$ imply $J_{t,t}(r) \leq J_{0,2t}(r)$. From this

$$\begin{aligned} (4.35) \quad G_\xi(0) &= -J_{t,t}^*(\xi) = \inf_{r \in \mathbb{R}} \{J_{t,t}(r) - \xi r\} \\ &\leq \inf_{r \in \mathbb{R}} \{J_{0,2t}(r) - \xi r\} = -J_{0,2t}^*(\xi) < \infty. \end{aligned}$$

Continuity at $a = 0$. To show that G_ξ is also continuous at the endpoints, we first obtain a lower bound. For any $r \in \mathbb{R}$,

$$J_{t-a,t}^*(\xi) \geq r\xi - J_{t-a,t}(r)$$

hence, by continuity of $J_{s,t}$ in the (s, t) argument,

$$(4.36) \quad \liminf_{a \rightarrow 0} J_{t-a,t}^*(\xi) \geq r\xi - J_{t,t}(r).$$

Supremum over r gives $\liminf_{a \rightarrow 0} J_{t-a,t}^*(\xi) \geq J_{t,t}^*(\xi)$.

For the upper bound, let $0 < a < t$. Varadhan's theorem (Thm. 4.3.1 in [12]) applies in the present setting. This is justified in the proof of Cor. 2.5 below and another similar justification is given for (5.4) below. Consequently

$$\begin{aligned}
 J_{t,t}^*(\xi) &= \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{\lfloor nt \rfloor, \lfloor nt \rfloor}} \\
 &\geq \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{\lfloor n(t-a) \rfloor, \lfloor nt \rfloor}} + \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \sum_{i=\lfloor n(t-a) \rfloor + 1}^{\lfloor nt \rfloor} \log Y_{i, \lfloor nt \rfloor}} \\
 (4.37) \quad &= J_{t-a,t}^*(\xi) + a \log \mathbb{E} Y^\xi.
 \end{aligned}$$

Taking $a \searrow 0$ yields continuity at $a = 0$.

Continuity at $a = t$. The lower bound follows as in the previous case. For the upper bound we use a path counting argument. Let $e^{nF(s,t)}$ be an upper bound on the number of paths in $\Pi_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ such that $F(0+, t) = 0$. Consider first the case where $0 \leq \xi < 1$. Then

$$\begin{aligned}
 J_{t-a,t}^*(\xi) &= \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} \left(\sum_{x \in \Pi_{(\lfloor n(t-a) \rfloor, \lfloor nt \rfloor)}} \prod_{i=1}^{\lfloor nt \rfloor + \lfloor n(t-a) \rfloor} Y_{x_i} \right)^\xi \\
 (4.38) \quad &\leq \lim_{n \rightarrow \infty} n^{-1} \log \sum_{x \in \Pi_{(\lfloor n(t-a) \rfloor, \lfloor nt \rfloor)}} \prod_{i=1}^{\lfloor nt \rfloor + \lfloor n(t-a) \rfloor} \mathbb{E}(Y)^\xi \\
 &= F(t-a, t) + (2-a/t) J_{0,t}^*(\xi)
 \end{aligned}$$

For $1 \leq \xi < \mu$ Jensen's inequality yields

$$(4.39) \quad J_{t-a,t}^*(\xi) \leq \xi F(t-a, t) + (2-a/t) J_{0,t}^*(\xi).$$

Let $a \nearrow t$ to get the continuity.

$G_\xi^{**} = G_\xi$ is a consequence of convexity and lower semicontinuity, by [28, Thm. 12.2] \square

Proof of Proposition 4.3. The remainder of the proof is convex analysis. The goal is to derive this formula for the right tail rate function $J_{s,t}$:

$$(4.40) \quad J_{s,t}(r) = \sup_{\xi \in [0, \mu)} \left\{ r\xi - \inf_{\theta \in (\xi, \mu)} (tM_\theta(\xi) - sM_{\mu-\theta}(-\xi)) \right\}.$$

We begin by expressing the explicitly known dual $R_s^*(\xi)$ from (4.20) in terms of the unknown function $J_{(s,t)-\bar{\nu}(a)}$. Equation (4.26) says that $H_{s,t}^a$ is the infimal convolution of κ_a and $J_{(s,t)-\bar{\nu}(a)}$, in symbols $H_{s,t}^a = \kappa_a \square J_{(s,t)-\bar{\nu}(a)}$. By Theorem 16.4 in [28] addition is dual to infimal convolution. Starting with (4.28) we have

$$\begin{aligned}
 R_s^*(\xi) &= \sup_{-t \leq a \leq s} \sup_{r \in \mathbb{R}} \{ r\xi - (\kappa_a \square J_{(s,t)-\bar{\nu}(a)})(r) \} \\
 (4.41) \quad &= \sup_{-t \leq a \leq s} (\kappa_a \square J_{(s,t)-\bar{\nu}(a)})^*(\xi) \\
 &= \sup_{-t \leq a \leq s} \{ \kappa_a^*(\xi) + J_{(s,t)-\bar{\nu}(a)}^*(\xi) \}.
 \end{aligned}$$

Combining this with (4.20) gives, for $0 \leq \xi < \theta$,

$$(4.42) \quad s \log \Gamma(\theta - \xi) - s \log \Gamma(\theta) = \sup_{-t \leq a \leq s} \{ \kappa_a^*(\xi) + J_{(s,t)-\bar{\nu}(a)}^*(\xi) \}.$$

Now regard $\xi \in [0, \mu]$ fixed, and let $\theta \in (\xi, \mu)$ vary. Introduce temporary definitions

$$(4.43) \quad u_a(\theta) = \begin{cases} -h_\xi(\theta) = M_{\mu-\theta}(-\xi) = \log \Gamma(\mu - \theta + \xi) - \log \Gamma(\mu - \theta), & -t \leq a \leq 0 \\ d_\xi(\theta) = M_\theta(\xi) = \log \Gamma(\theta - \xi) - \log \Gamma(\theta), & 0 < a \leq s. \end{cases}$$

Substitute (4.22) and (4.43) into equation (4.42) to get

$$(4.44) \quad s \log \frac{\Gamma(\theta - \xi)}{\Gamma(\theta)} - t \log \frac{\Gamma(\mu - \theta + \xi)}{\Gamma(\mu - \theta)} = \sup_{-t \leq a \leq s} \{a u_a(\theta) + J_{(s,t)-\bar{\mathbf{v}}(a)}^*(\xi)\}$$

The right-hand side begins to resemble a convex dual, and will allow us to solve for $J_{s,t}$. We can specialize to the case $s = t$ because $(t, t) - \bar{\mathbf{v}}(a)$ gives all the pairs (s, t) with $0 \leq s \leq t$. When $s = t$, the $J_{s,t} = J_{t,s}$ symmetry allows us to write (4.44) as

$$t(d_\xi(\theta) + h_\xi(\theta)) = \sup_{0 \leq a \leq t} \{a(h_\xi(\theta) \vee d_\xi(\theta)) + J_{t-a,t}^*(\xi)\}$$

and it splits into cases as follows:

$$t(d_\xi(\theta) + h_\xi(\theta)) = \begin{cases} \sup_{0 \leq a \leq t} \{a h_\xi(\theta) + J_{t-a,t}^*(\xi)\}, & \theta \in [(\mu + \xi)/2, \mu) \\ \sup_{0 \leq a \leq t} \{a d_\xi(\theta) + J_{t-a,t}^*(\xi)\}, & \theta \in (\xi, (\mu + \xi)/2]. \end{cases}$$

We can discard one of the branches above. For if $\theta' = \mu + \xi - \theta$ then $d_\xi(\theta') = h_\xi(\theta)$ and we see that the two equations given by the two branches are in fact equivalent. So we restrict to the case $\theta \in [(\mu + \xi)/2, \mu)$ and continue with

$$(4.45) \quad t(d_\xi(\theta) + h_\xi(\theta)) = \sup_{0 \leq a \leq t} \{a h_\xi(\theta) + J_{t-a,t}^*(\xi)\}.$$

The function h_ξ is strictly increasing so we can change variables via $v = h_\xi(\theta)$ between the intervals $\theta \in [(\mu + \xi)/2, \mu)$ and $v \in [h_\xi((\mu + \xi)/2), \infty)$. Recall also $G_\xi(a) = -J_{t-a,t}^*(\xi)$ from Lemma 4.6. This turns (4.45) into

$$(4.46) \quad t((d_\xi \circ h_\xi^{-1})(v) + v) = \sup_{0 \leq a \leq t} \{a v - G_\xi(a)\} = G_\xi^*(v), \quad h_\xi(\frac{\mu + \xi}{2}) \leq v < \infty.$$

Utilizing $G_\xi = G_\xi^{**}$ we get an expression for the rate function J :

$$(4.47) \quad \begin{aligned} J_{t-a,t}(r) &= \sup_{\xi \in [0, \mu]} \{r\xi - J_{t-a,t}^*(\xi)\} = \sup_{\xi \in [0, \mu]} \{r\xi + G_\xi(a)\} \\ &= \sup_{\xi \in [0, \mu]} \{r\xi + \sup_{v \in \mathbb{R}} [a v - G_\xi^*(v)]\} \\ &= \sup_{\xi \in [0, \mu]} \{r\xi + \sup_{v \in [h_\xi((\mu + \xi)/2), \infty)} [a v - G_\xi^*(v)]\} \\ (4.48) \quad &= \sup_{\xi \in [0, \mu]} \{r\xi + \sup_{v \in [h_\xi((\mu + \xi)/2), \infty)} [(a - t)v - t d_\xi(h_\xi^{-1}(v))]\}. \end{aligned}$$

In the next to last equality above we restricted the supremum over v to the interval $v \in [h_\xi((\mu + \xi)/2), \infty)$. This is justified because G_ξ^* is convex, $a \geq 0$, and from (4.46) we can compute the right derivative $(G_\xi^*)'(h_\xi(\frac{\mu + \xi}{2}) +) = 0$. The restriction of the supremum then allows us to replace $G_\xi^*(v)$ with (4.46).

The proof is complete. In the case $0 < s \leq t$ take $a = t - s$ on line (4.47). Line (4.48) is the desired representation for $J_{s,t}$. It turns into (4.40) by the v to θ change of variable. The case $s > t$ follows from the symmetry $J_{s,t}(r) = J_{t,s}(r)$. \square

The next lemma makes explicit the formula(s) for $J_{s,t}^*$ that were implicit in the proof of Proposition 4.3.

LEMMA 4.7. *Let $s, t \geq 0$ and $\xi \in [0, \mu]$. Then*

$$(4.49) \quad J_{s,t}^*(\xi) = \inf_{\rho \in (\xi, \mu)} \{tM_\rho(\xi) - sM_{\mu-\rho}(-\xi)\}$$

$$(4.50) \quad = \inf_{\theta \in (\xi, \mu)} \{sM_\theta(\xi) - tM_{\mu-\theta}(-\xi)\}.$$

Proof. (4.50) comes from (4.49) by the change of variable $\rho = \mu + \xi - \theta$. Comparison of the two shows that we can assume $s \leq t$. To prove (4.49) for $s \leq t$, start from Lemma 4.6:

$$J_{s,t}^*(\xi) = -G_\xi(t - s) = -G_\xi^{**}(t - s) = -\sup_{v \in \mathbb{R}} \{(t - s)v - G_\xi^*(v)\}.$$

Restrict the supremum as in (4.47)–(4.48), substitute in (4.46) and change variables from v to $\theta = h_\xi^{-1}(v)$. \square

Proof of Corollary 2.5. If $\xi \geq \mu$,

$$\xi \log Z_{[ns], [nt]} \geq \sum_j \xi \log Y_{x_j}$$

for any particular path $x. \in \Pi_{[ns], [nt]}$, and then $\Lambda_{s,t}(\xi) = \infty$ comes from $M_\mu(\xi) = \infty$ from (2.2).

Let $\xi < \mu$. Pick $\gamma > 1$ such that $\gamma\xi < \mu$. Then the bound

$$\sup_n n^{-1} \log \mathbb{E} e^{\gamma\xi \log Z_{[ns], [nt]}} < \infty$$

follows from pathcounting, as in (4.38)–(4.39). This bound is sufficient for Varadhan's theorem (Thm. 4.3.1 in [12]) which gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \Lambda_{s,t}(\xi) &= n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns], [nt]}} = I_{s,t}^*(\xi) = \sup_{r \in \mathbb{R}} \{r\xi - I_{s,t}(r)\} \\ &= \sup_{r \geq p(s,t)} \{r\xi - I_{s,t}(r)\} = \sup_{r \geq p(s,t)} \{r\xi - J_{s,t}(r)\}. \end{aligned}$$

We discarded $\{I_{s,t} = \infty\} = \{r < p(s,t)\}$ from the supremum. Since $I_{s,t}$ increases for $r \geq p(s,t)$, the case $\xi \leq 0$ of (2.11) follows. For $\xi \geq 0$ the values $J_{s,t}(r) = 0$ for $r < p(s,t)$ can be put back in because they do not alter the supremum. Consequently $\Lambda_{s,t}(\xi) = J_{s,t}^*(\xi)$ for $\xi \geq 0$. Lemma 4.7 completes the proof of this corollary. \square

There is nothing new in the proof of Corollary 2.8 so we omit it.

5. PROOFS FOR THE STATIONARY LOG-GAMMA MODEL

In this section we prove the results of Section 2.2.

Proof of Theorem 2.14. Coarse-graining and simple error bounds readily give this limit:

$$\begin{aligned}
p^{(\theta),\text{hor}}(s,t) &= \lim_{n \rightarrow \infty} n^{-1} \log Z_{[ns],[nt]}^{(\theta),\text{hor}} \\
&= \lim_{n \rightarrow \infty} \max_{1 \leq k \leq [ns]} \left(n^{-1} \sum_{i=1}^k \log U_{i,0} + n^{-1} \log Z_{(k,1),([ns],[nt])}^{\square} \right) \\
&= \sup_{0 \leq a \leq s} \{-a\Psi_0(\theta) + p(s-a, t)\} \\
&= \sup_{0 \leq a \leq s} \inf_{0 < \rho < \mu} \{-a\Psi_0(\theta) + (a-s)\Psi_0(\rho) - t\Psi_0(\mu-\rho)\}.
\end{aligned}$$

In the last step we substituted in (2.5). Formula (2.30) follows from this by some calculus.

From the definition (2.22) of $Z_{[ns],[nt]}^{(\theta),\text{hor}}$ follow inequalities analogous to (4.17), and then with arguments like those in the proof of Lemma 4.5 we derive a right tail LDP

$$(5.1) \quad \lim_{n \rightarrow \infty} n^{-1} \log \mathbb{P}\{Z_{[ns],[nt]}^{(\theta),\text{hor}} \geq nr\} = -J_{\theta,\text{hor}}(r) = - \inf_{a \in [0,s]} (R_a \square J_{s-a,t})(r)$$

where R_a is the rate function from (4.18). For $\xi \geq 0$ the l.m.g.f. in (2.24) satisfies $\Lambda_{\theta,(s,t)}^{\text{hor}}(\xi) = J_{\theta,\text{hor}}^*(\xi)$. This would be a consequence of Varadhan's theorem if we had a full LDP, but now we have to justify this separately and we do so in Lemma 5.1 below. 1 Proceeding as in (4.41) and using (4.50)

$$\begin{aligned}
\Lambda_{\theta,(s,t)}^{\text{hor}}(\xi) &= \sup_{a \in [0,s]} (R_a^*(\xi) + J_{s-a,t}^*(\xi)) \\
&= \sup_{a \in [0,s]} \inf_{\rho \in (\xi, \mu)} \{aM_{\theta}(\xi) + (s-a)M_{\rho}(\xi) - tM_{\mu-\rho}(-\xi)\}.
\end{aligned}$$

Formula (2.31) follows from some calculus. The sup and inf can be interchanged by a minimax theorem (see for example [19]) and this makes the calculus easier. \square

LEMMA 5.1. *Let $Z_{[ns],[nt]}^{(\theta),\text{hor}}$ the partition function given by (2.22) and let $J_{\theta,\text{hor}}(r)$ as given by (5.1). Then for $0 \leq \xi < \theta$*

$$\lim_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{hor}}} = \sup_{r \in \mathbb{R}} \{r\xi - J_{\theta,\text{hor}}(r)\} = J_{\theta,\text{hor}}^*(\xi).$$

Proof. Let $0 < \xi < \theta$. Set

$$\underline{\gamma} = \varliminf_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{hor}}} \quad \text{and} \quad \overline{\gamma} = \varlimsup_{n \rightarrow \infty} n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{hor}}}.$$

First an exponential Chebyshev argument for a lower bound:

$$n^{-1} \log \mathbb{P}\{\log Z_{[ns],[nt]}^{(\theta),\text{hor}} \geq nr\} \leq -\xi r + n^{-1} \log \mathbb{E} e^{\xi \log Z_{[ns],[nt]}^{(\theta),\text{hor}}}.$$

Letting $n \rightarrow \infty$ along a suitable subsequence gives $\underline{\gamma} \geq \xi r - J_{\theta,\text{hor}}(r)$ for all $r \in \mathbb{R}$. Thus $\underline{\gamma} \geq J_{\theta,\text{hor}}^*(\xi)$ holds.

For the upper bound we claim that

$$(5.2) \quad \lim_{r \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \mathbf{1}_{\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr\}} \right) = -\infty.$$

Assume for a moment that (5.2) holds. To establish the upper bound let $\delta > 0$ and partition \mathbb{R} with $r_i = i\delta$, $i \in \mathbb{Z}$.

$$(5.3) \quad n^{-1} \log \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \right) \leq n^{-1} \log \left[\sum_{i=-m}^m e^{n\xi r_{i+1}} \mathbb{P}\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr_i\} + e^{n\xi r_{-m}} + \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \mathbf{1}_{\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr_m\}} \right) \right].$$

By (5.2), for each $M > 0$ there exists $m = m(M)$ so that

$$n^{-1} \log \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \mathbf{1}_{\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr_m\}} \right) < -M.$$

A limit along a suitable subsequence in (5.3) yields

$$\begin{aligned} \overline{\gamma} &\leq \max_{-m \leq i \leq m} \{\xi r_{i+1} - J_{\theta, \text{hor}}(r_i)\} \vee \xi r_{-m} \vee (-M) \\ &\leq \left(\sup_{r \in \mathbb{R}} \{\xi r - J_{\theta, \text{hor}}(r)\} + \xi \delta \right) \vee \xi r_{-m} \vee (-M). \end{aligned}$$

The proof of the lemma follows by letting $\delta \rightarrow 0$, $m \rightarrow \infty$ and $M \rightarrow \infty$.

Now to show (5.2). Note that there exists $\alpha > 1$ such that $\alpha\xi < \theta$,

$$(5.4) \quad \sup_n \left(\mathbb{E} e^{\alpha\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \right)^{1/n} < \infty.$$

To see this, distinguish cases where $\alpha\xi < 1$ or otherwise. Let N denote the number of paths and recall that $N \leq e^{cn}$ for some $c > 0$: For $\alpha\xi < 1$,

$$\begin{aligned} \left(\mathbb{E} e^{\alpha\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \right)^{1/n} &= \left(\mathbb{E} \left[\left(\sum_{x \in \Pi_{([ns], [nt])}} \prod_{i=1}^{[ns]+[nt]} Y_{x_j} \right)^{\alpha\xi} \right] \right)^{1/n} \leq \left(N \prod_{i=1}^{[nt]+[ns]} \mathbb{E} Y^{\alpha\xi} \right)^{1/n} \\ &\leq e^c M_\theta (\alpha\xi)^{t+s}. \end{aligned}$$

For $\alpha\xi \geq 1$, Jensen's inequality gives

$$\begin{aligned} \left(\mathbb{E} e^{\alpha\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \right)^{1/n} &= \left(\mathbb{E} \left[\left(\sum_{x \in \Pi_{([ns], [nt])}} \prod_{i=1}^{[ns]+[nt]} Y_{x_j} \right)^{\alpha\xi} \right] \right)^{1/n} \leq \left(N^{\alpha\xi} \prod_{i=1}^{[nt]+[ns]} \mathbb{E} Y^{\alpha\xi} \right)^{1/n} \\ &\leq e^{c\alpha\xi} M_\theta (\alpha\xi)^{t+s}. \end{aligned}$$

To show (5.2), use Hölder's inequality,

$$\begin{aligned} n^{-1} \log \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \mathbf{1}_{\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr\}} \right) \\ \leq \alpha^{-1} \log \sup_n \left(\mathbb{E} e^{\alpha\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \right)^{1/n} + (\alpha - 1) \alpha^{-1} n^{-1} \log \mathbb{P}\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr\}. \end{aligned}$$

Taking a limit $n \rightarrow \infty$ we conclude

$$(5.5) \quad \overline{\lim}_{n \rightarrow \infty} n^{-1} \log \mathbb{E} \left(e^{\xi \log Z_{[ns], [nt]}^{(\theta), \text{hor}}} \mathbf{1}_{\{\log Z_{[ns], [nt]}^{(\theta), \text{hor}} \geq nr\}} \right) \leq C_1 - C_2 J_{\theta, \text{hor}}(r),$$

for positive constants C_1, C_2 . Letting $r \rightarrow \infty$ finishes the proof because $\lim_{r \rightarrow \infty} J_{\theta, \text{hor}}(r) = \infty$. \square

Proof of Theorem 2.11. We can assume $0 < \xi < \theta \wedge (\mu - \theta)$ because otherwise the boundary variables alone force the l.m.g.f. to blow up.

Let us record the counterpart of (2.31) for $Z_{[ns], [nt]}^{(\theta), \text{hor}}$. Condition (2.28) becomes

$$(5.6) \quad t(\Psi_0(\mu - \theta) - \Psi_0(\mu - \theta - \xi)) \geq s(\Psi_0(\theta + \xi) - \Psi_0(\theta)).$$

The conclusion becomes that the limit in (2.25) exists and is given by

$$(5.7) \quad \Lambda_{\theta, (s, t)}^{\text{ver}}(\xi) = \begin{cases} tM_{\mu - \theta}(\xi) - sM_{\theta}(-\xi), & \text{if (5.6) holds} \\ \Lambda_{t, s}(\xi) = \Lambda_{s, t}(\xi), & \text{if (5.6) fails.} \end{cases}$$

The logarithmic limits lead to the formula

$$(5.8) \quad \Lambda_{\theta, (s, t)}(\xi) = \Lambda_{\theta, (s, t)}^{\text{hor}}(\xi) \vee \Lambda_{\theta, (s, t)}^{\text{ver}}(\xi)$$

and we need to justify that this is the same as the maximum in (2.20). This comes from several observations.

(i) $\Lambda_{s, t}(\xi) = J_{s, t}^*(\xi)$ is always bounded above by the first branches of both (2.31) or (5.7). This is evident from equations (4.49)–(4.50).

(ii) Conditions (2.28) and (5.6) together define three ranges for (s, t) :

- (a) (2.28) and (5.6) both hold iff $\alpha_1 t \leq s \leq \alpha_2 t$;
- (b) (2.28) holds and (5.6) fails iff $s > \alpha_2 t$;
- (c) (2.28) fails and (5.6) holds iff $s < \alpha_1 t$.

The constants $0 < \alpha_1 < \alpha_2$ can be read off (2.28) and (5.6) and the strict inequalities justified by the strict concavity of Ψ_0 .

(iii) In the maximum in (2.20) we have

$$(5.9) \quad sM_{\theta}(\xi) - tM_{\mu - \theta}(-\xi) \geq tM_{\mu - \theta}(\xi) - sM_{\theta}(-\xi)$$

iff $s \geq \alpha_3 t$ for a constant $\alpha_3 > 0$ that can be read off from above. Strict concavity of Ψ_0 implies that $0 < \alpha_1 < \alpha_3 < \alpha_2$.

Now we argue that

$$(5.10) \quad \Lambda_{\theta, (s, t)}(\xi) = \max \{ sM_{\theta}(\xi) - tM_{\mu - \theta}(-\xi), tM_{\mu - \theta}(\xi) - sM_{\theta}(-\xi) \}.$$

This is clear in case (a) as this maximum is exactly $\Lambda_{\theta, (s, t)}^{\text{hor}}(\xi) \vee \Lambda_{\theta, (s, t)}^{\text{ver}}(\xi)$. In case (b), $\Lambda_{\theta, (s, t)}^{\text{hor}}(\xi)$ equals the left-hand side of (5.9) which dominates both the right-hand side of (5.9) and $\Lambda_{s, t}(\xi)$. Consequently in case (b) also (5.8) is the same as (5.10). Case (c) is symmetric to (b). This completes the proof of (5.10).

With one additional observation we can verify Remark 2.15. Namely, $\Lambda_{s, t}(\xi)$ is in fact *strictly* bounded above by the first branch of either (2.31) or (5.7). The claim is easily verifiable when either of conditions (b), (c) are in effect. To see the strict domination when (a) holds, note that the unique minimizers in formulas (4.49)–(4.50) are linked by $\rho = \mu + \xi - \theta$. But if these formulas matched both first branches in (2.31) and (5.7), the connection would have to be $\rho = \mu - \theta$. This together with (5.10) implies that $\Lambda_{s, t}(\xi) < \Lambda_{\theta, (s, t)}(\xi)$ for all $\theta \in (0, \mu)$. \square

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